L'HÔPITAL'S RULE:

(First appear in L'Hopital's introductory calculus text back in 1696, "Analyse des Infiniment Petits" Actually BERNOULLI IS MORE LIKELY THE DISCOVERER OF THE RULE)

Def$^{\underline{n}}$/ Inderminant Forms:

1.) $\lim\limits_{x\to a}\left(\frac{f(x)}{g(x)}\right)$ is of type $\left(\frac{0}{0}\right)$ if $\lim\limits_{x\to a}(f(x))=0$ AND $\lim\limits_{x\to a}(g(x))=0$

2.) $\lim\limits_{x\to a}\left(\frac{f(x)}{g(x)}\right)$ is of type $\left(\frac{\infty}{\infty}\right)$ if $\lim\limits_{x\to a}(f(x))=\infty$ AND $\lim\limits_{x\to a}(g(x))=\infty$

We say the same for one-sided limits and for $a=\pm\infty$. Other indeterminant forms like $\infty-\infty$, $0\cdot\infty$, 0^0, 1^∞, ∞^0 are defined similarly

L'Hôpital's Rule

If $\lim\limits_{x\to a}\left(\frac{f(x)}{g(x)}\right)$ is type $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$ and f and g are differentiable near $x=a$ then we have that

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right)=\lim_{x\to a}\left(\frac{f'(x)}{g'(x)}\right)$$

Notation I like to use to clarify the application of L'Hopital's Rule is

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right)\underset{\left(\frac{0}{0}\right)}{\overset{\mathcal{L}}{=}}\lim_{x\to a}\left(\frac{f'(x)}{g'(x)}\right) \quad \text{OR} \quad \lim_{x\to a}\left(\frac{f(x)}{g(x)}\right)\underset{\left(\frac{\infty}{\infty}\right)}{\overset{\mathcal{L}}{=}}\lim_{x\to a}\left(\frac{f'(x)}{g'(x)}\right)$$

$\boxed{E1}$ Notice $\lim\limits_{x\to 0}(\sin(x))=\sin(0)=0$ and $\lim\limits_{x\to 0}(x)=0$ thus

$$\lim_{x\to 0}\left(\frac{\sin(x)}{x}\right)\underset{\left(\frac{0}{0}\right)}{\overset{\mathcal{L}}{=}}\lim_{x\to 0}\left(\frac{\cos(x)}{1}\right)=\cos(0)=\boxed{1}.$$

L'Hopital's Rule removed the indeterminancy. We "proved" this back on 33 using a geometric argument.

$\boxed{E2}$ This one is type $\frac{\infty}{\infty}$, we'll apply L'Hopital's Rule twice,

$$\lim_{x\to\infty}\left(\frac{x^2+x-2}{x^2+3}\right)\underset{\left(\frac{\infty}{\infty}\right)}{\overset{\mathcal{L}}{=}}\lim_{x\to\infty}\left(\frac{2x+1}{2x}\right)\underset{\left(\frac{\infty}{\infty}\right)}{\overset{\mathcal{L}}{=}}\lim_{x\to\infty}\left(\frac{2}{2}\right)=\boxed{1}$$

• <u>Remark</u>: We do not use the quotient rule. We differentiate the numerator and denominator seperately than take their quotient.

Proof of L'Hopital's Rule (Actually its JOHN BERNOULLI'S RULE)

Let $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right)$ be of type $\left(\frac{0}{0}\right)$ and let $f'(a)$ and $g'(a)$ exist with $g'(a) \neq 0$. We'll prove the rule in this simple case, the general proof can be found in Appendix G of Thomas' 10^{th} Ed. of "CALCULUS". Recall by the very definition of $f'(a)$ and $g'(a)$ we find,

$$f'(a) = \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right) = \lim_{x\to a}\left(\frac{f(x)}{x-a}\right) - \lim_{x\to a}\left(\frac{\cancel{f(a)}^{\,0}}{x-a}\right)$$

$$g'(a) = \lim_{x\to a}\left(\frac{g(x)-g(a)}{x-a}\right) = \lim_{x\to a}\left(\frac{g(x)}{x-a}\right) - \lim_{x\to a}\left(\frac{\cancel{g(a)}^{\,0}}{x-a}\right)$$

Notice in the above it is crucial that $f(a)=0$ and $g(a)=0$, so that

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \lim_{x\to a}\left(\frac{\frac{f(x)}{x-a}}{\frac{g(x)}{x-a}}\right) = \frac{\lim_{x\to a}\left(\frac{f(x)}{x-a}\right)}{\lim_{x\to a}\left(\frac{g(x)}{x-a}\right)} = \frac{f'(a)}{g'(a)} \quad //$$

E3

$$\lim_{x\to\infty}\left(\frac{\ln(x^2)}{\sqrt[3]{x}}\right) \underset{\left(\frac{\infty}{\infty}\right)}{\overset{\mathcal{L}}{=}} \lim_{x\to\infty}\left(\frac{\frac{2}{x}}{\frac{1}{3}x^{-2/3}}\right) = \lim_{x\to\infty}\left(\frac{6x^{2/3}}{x}\right) = \lim_{x\to\infty}\left(\frac{6}{\sqrt[3]{x}}\right) = \boxed{0}$$

E4 These ideas also make sense for one-sided limits. For example,

$$\lim_{x\to 0^+}\left(xe^{\frac{1}{x}}\right) = \lim_{x\to 0^+}\left(\frac{e^{\frac{1}{x}}}{\frac{1}{x}}\right)$$

Notice we had to rewrite the intial expression to make it have the form $\left(\frac{\infty}{\infty}\right)$.

$$\underset{\left(\frac{\infty}{\infty}\right)}{\overset{\mathcal{L}}{=}} \lim_{x\to 0^+}\left(\frac{e^{\frac{1}{x}}\left(\frac{-1}{x^2}\right)}{\frac{-1}{x^2}}\right)$$

$$= \lim_{x\to 0^+}\left(e^{\frac{1}{x}}\right)$$

$$= e^{\left(\lim_{x\to 0^+}\left(\frac{1}{x}\right)\right)}$$

$$= e^{\infty} = \boxed{\infty}$$

Remark: In $\boxed{E4}$ we had type $(0 \cdot \infty)$ to begin. We rewrote it so that 0 became $\frac{1}{1/0} = \frac{1}{\infty}$ meaning $(0 \cdot \infty) \longrightarrow (\frac{\infty}{\infty})$ which is what we need for L'Hôpital's to apply. Generally

$$\underset{(\text{type } 0 \cdot \infty)}{\lim (fg)} \quad \text{with} \quad \begin{array}{l} \lim f = 0 \\ \lim g = \infty \end{array} \rightsquigarrow \underset{(\text{type } \frac{\infty}{\infty})}{\lim \left(\frac{g}{1/f}\right)}$$

$\boxed{E5}$

$$\lim_{x \to \infty} \left(e^{-x} x^2\right) = \lim_{x \to \infty} \left(\frac{x^2}{e^x}\right)$$

$$\underset{(\frac{\infty}{\infty})}{\overset{\mathcal{L}}{=}} \lim_{x \to \infty} \left(\frac{2x}{e^x}\right)$$

$$\underset{(\frac{\infty}{\infty})}{\overset{\mathcal{L}}{=}} \lim_{x \to \infty} \left(\frac{2}{e^x}\right) = \boxed{0}$$

Notice the exponential grows much quicker that x^2 as $x \longrightarrow \infty$. In fact its easy to see even if x^2 is replaced with x^n we will find the same result. Exponential growth is faster than polynomial growth.

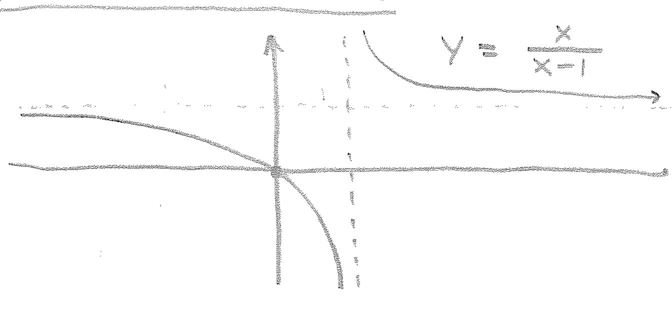
$\boxed{E6}$ Logarithmic growth is slower than polynomial growth,

$$\lim_{x \to \infty} \left(\frac{x^2}{\ln(x)}\right) \underset{(\frac{\infty}{\infty})}{\overset{\mathcal{L}}{=}} \lim_{x \to \infty} \left(\frac{2x}{\frac{1}{x}}\right) = \lim_{x \to \infty} \left(2x^2\right) = \boxed{\infty}$$

$\boxed{E7}$ INCORRECT EXAMPLE, WHAT NOT TO DO.

$$\lim_{x \to 1} \left(\frac{x}{x-1}\right) \overset{\mathcal{L}}{=} \lim_{x \to 1} \left(\frac{1}{1}\right) = 1. \qquad (\text{FALSE!})$$

WHAT DID I DO WRONG?



As you can see from graph

$$\lim_{x \to 1} \left(\frac{x}{x-1}\right) \text{ d.n.e.}$$

E8 Here is type $(\infty - \infty)$, see how we handle it, (see pg. 5 graphs)

$$\lim_{\theta \to 0^+} \left(\csc(\theta) - \cot(\theta) \right) = \lim_{\theta \to 0^+} \left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta} \right)$$

$$= \lim_{\theta \to 0^+} \left(\frac{1 - \cos\theta}{\sin\theta} \right)$$

they had a common denom. it was just hidden by notation.

$$\underset{(\frac{0}{0})}{\overset{H}{=}} \lim_{\theta \to 0^+} \left(\frac{\sin\theta}{\cos\theta} \right)$$

$$= \lim_{\theta \to 0^+} \left(\tan\theta \right)$$

$$= \tan(0) = \boxed{0}$$

E9 Lets try another $(\infty - \infty)$ example, (#30 on pg. 305 of Stewarts 2nd Ed.)

$$\lim_{x \to 1} \left(\frac{1}{\ln(x)} - \frac{1}{x-1} \right) = \lim_{x \to 1} \left(\frac{(x-1) - \ln(x)}{\ln(x)\,(x-1)} \right)$$

made a common denominator.

$$\underset{(\frac{0}{0})}{\overset{H}{=}} \lim_{x \to 1} \left(\frac{1 - \frac{1}{x}}{\frac{1}{x}(x-1) + \ln(x)} \right)$$

: used product rule on denom.

$$= \lim_{x \to 1} \left(\frac{1 - \frac{1}{x}}{1 - \frac{1}{x} + \ln(x)} \right)$$

$$\underset{(\frac{0}{0})}{\overset{H}{=}} \lim_{x \to 1} \left(\frac{\frac{1}{x^2}}{\frac{1}{x^2} + \frac{1}{x}} \right)$$

$$= \frac{1}{1+1} = \boxed{\frac{1}{2}}$$

INDETERMINANT POWERS

1.) $\lim f = 0$ and $\lim g = 0$ then $\lim (f)^g$ is type (0^0)

2.) $\lim f = \infty$ and $\lim g = 0$ then $\lim (f)^g$ is type $(\infty)^0$

3.) $\lim f = 1$ and $\lim g = \pm\infty$ then $\lim (f)^g$ is type (1^∞)

To resolve these indeterminancies we'll need to employ a little trick,

$$\boxed{[f(x)]^{g(x)} = e^{\ln\left([f(x)]^{g(x)}\right)} = e^{g(x)\ln(f(x))}}$$

Additionally because the exponential function is continous everywhere we can pass limits inside it,

$$\lim_{x\to a}\left([f(x)]^{g(x)}\right) = \lim_{x\to a}\left(e^{g(x)\ln(f(x))}\right) = e^{\lim_{x\to a}(g(x)\ln(f(x)))}$$

We use this idea throughout the next few examples we replace the original problem $\lim_{x\to a}\left([f(x)]^{g(x)}\right)$ with the problem of finding $\lim_{x\to a}(g(x)\ln(f(x)))$ then what happens is:

1.) $\lim_{x\to a}(g(x)\ln(f(x))) = b \in \mathbb{R} \Rightarrow \lim_{x\to a}(f(x))^{g(x)} = e^{\lim_{x\to a}(g\ln f)} = e^b$

2.) $\lim_{x\to a}(g(x)\ln(f(x))) = \infty \Rightarrow \lim_{x\to a}(f(x))^{g(x)} = e^{\lim_{x\to a}(g\ln f)} = e^\infty = \infty$

3.) $\lim_{x\to a}(g(x)\ln(f(x))) = -\infty \Rightarrow \lim_{x\to a}(f(x))^{g(x)} = e^{\lim_{x\to a}(g\ln f)} = e^{-\infty} = 0$

- Now go work thru the examples, then come back and read this and see if it makes more sense.

E10 (Type 0^0)

$$\lim_{x\to 0^+}(x^x) = \lim_{x\to 0^+}\left(e^{x\ln(x)}\right) = e^{\underbrace{\lim_{x\to 0^+}(x\ln(x))}_{(*)}}$$

$$(*)\ \lim_{x\to 0^+}(x\ln(x)) = \lim_{x\to 0^+}\left(\frac{\ln(x)}{\frac{1}{x}}\right) \underset{\left(\frac{\infty}{\infty}\right)}{\overset{L}{=}} \lim_{x\to 0^+}\left(\frac{\frac{1}{x}}{\frac{-1}{x^2}}\right) = \lim_{x\to 0^+}(-x) = 0.$$

Therefore using $(*)$ we get $\lim_{x\to 0^+}(x^x) = e^0 = \boxed{1}$

$\boxed{E11}$ (TYPE 1^{∞}) # 34 on pg. 305,

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nx} = \lim_{n\to\infty}\left(e^{nx\ln(1+\frac{1}{n})}\right)$$

$$= e^{\underbrace{\lim_{n\to\infty}(nx\ln(1+\frac{1}{n}))}_{(*)}}$$

Now we need to calculate (*).

$$\lim_{n\to\infty}\left(nx\ln(1+\tfrac{1}{n})\right) = x\lim_{n\to\infty}\left(\frac{\ln(1+\frac{1}{n})}{\frac{1}{n}}\right)$$

why can I pull x out of the limit here?

$$\underset{(\frac{0}{0})}{\overset{LH}{=}} x\lim_{n\to\infty}\left(\frac{\frac{1}{1+1/n}\left(\frac{-1}{n^2}\right)}{\frac{-1}{n^2}}\right)$$

$$= x\lim_{n\to\infty}\left(\frac{1}{1+\underbrace{\tfrac{1}{n}}_{0}}\right)$$

$$= x$$

Thus we find $\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nx} = e^{x}}$. By the way some other texts define e by this limit; $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}$ this is clearly equivalent to our definition on pg. 32.

$\boxed{E12}$ Type (∞^{0})

$$\lim_{x\to\infty}\left(x^{\frac{1}{x}}\right) = \lim_{x\to\infty}\left(e^{\frac{1}{x}\ln(x)}\right) = e^{\underbrace{\lim_{x\to\infty}\left(\frac{\ln(x)}{x}\right)}_{*}}$$

Again we must find *,

$$\lim_{x\to\infty}\left(\frac{\ln(x)}{x}\right) \underset{(\frac{\infty}{\infty})}{\overset{LH}{=}} \lim_{x\to\infty}\left(\frac{\frac{1}{x}}{1}\right) = 0$$

Therefore,

$$\boxed{\lim_{x\to\infty}\left(x^{x}\right) = e^{0} = 1}$$