- integrals over G
- bilinearity

Lie Groups Final Exam Fall 2006

(1) Let V be a finite-dimensional representation space of a compact Lie group G.
(a) Show in detail that there is a G-invariant inner product on V. Check all the relevant properties.
(b) Show that if W is a G-submodule of V then $V = W \oplus U$ for some submodule U of V. Also show that V is a direct of irreducible submodules.

(2) Let G be any Lie group and V a finite-dimensional irreducible G-module. Show that every morphism $f : V \to V$ has the form $f = \lambda id_V$ for some $\lambda \in \mathbb{C}$.

(3) Using the fact that for irreducible V and $f \in Hom(V, V)$

$$\int_G (g \cdot f) dg = (\frac{1}{dim_{\mathbb{C}} V}) Tr(f) id_V$$

prove the following:
(a)

$$\int_G \phi(gf(g^{-1}v)) dg = (\frac{1}{dim_{\mathbb{C}} V}) Tr(f) \phi(v)$$

for $\phi \in V^*, f \in Hom(V, V)$, and $v \in V$
(b)

$$\int_G < gf(g^{-1}v), w > dg = (\frac{1}{dim_{\mathbb{C}} V}) Tr(f) < v, w >$$

for $f \in Hom(V, V), v, w \in V$
(c)

$$\int_G < g^{-1}v, \alpha >< g\beta, w > dg = (\frac{1}{dim_{\mathbb{C}} V}) < \beta, \alpha >< v, w >$$

for $\alpha, \beta, v, w \in V$. Hint: choose $f(u) =< u, \alpha > \beta$.

THE REST OF THE EXAM IS ON THE BACK OF THE PAGE

(4) Let V be a finite-dimensional representation space of a compact Lie group G endowed with a G-invariant inner product. Let $g \to (r_{ij}^V(g))$ denote the corresponding matrix representation relative to an orthonormal basis of V.
(a) Show that

$$\int_G r_{ij}^V(g)\overline{r_{kl}^V(g)}dg = (\frac{1}{dim_{\mathbb{C}}V})\delta_{ik}\delta_{jl}$$

(b) Define what it means to say f is a representative function of G and show that the functions r_{ij}^V are representative functions. What does the Peter Weyl theorem tell us about these functions?

(5) Show that a representation is determined up to isomorphism by its character.

$(R_g f)(x) \equiv f(xg)$

f_0 a representative fnct $\Rightarrow$ $f_0 \cdot G$ finite dim'l.

$C^0(G)$ & $L^2(G)$ dense with rep. fncts.