Definition Let G be a Lie group, then $C^0(G,\mathbb{K})$ denotes the space of all continuous maps $f: G \to \mathbb{K}$ such that the support of f, supp$f$, is compact. $C^0(G,\mathbb{K})$ is linear over $\mathbb{K}$ and is both a normed space relative to the sup norm and an inner product space as defined below. Thus for $f, g \in C^0(G,\mathbb{K})$, $\lambda \in \mathbb{K}$

$$(f+g)(x) = f(x)+g(x), \quad (\lambda f)(x) = \lambda f(x), \quad \forall x \in G$$

$$|f| = \sup_{x\in G} |f(x)|, \quad \langle f, g \rangle = \int_G f(x)\,\overline{g(x)}\,dx.$$

$(C^0(G,\mathbb{K}), +, \cdot, |\ |)$ is a Banach space. The completion of $C^0(G,\mathbb{K})$ relative to the inner product is a Hilbert space which we denote by $L^2(G)$

Proposition Let G be a compact Lie group and V a finite-dimensional representation space of G. Let $\{e_i\}$ denote an orthonormal basis of V relative to a G-invariant inner product on V. Let $r_{ij} : G \to \mathbb{K}$ denote the components of the corresponding matrix representation defined by

$$g \cdot e_i = \sum_j r_{ji}(g)\, e_j \qquad \forall g \in G$$

Let $s = s_V$ be the mapping from $V^* \otimes_{\mathbb{K}} V$ into $C^0(G,\mathbb{K})$ defined by

$$s(\alpha \otimes v)(g) = \alpha(g \cdot v) \qquad \forall \alpha \in V^*, v \in V, g \in G$$

Then

(1) The image of s, which we denote by $S(V)$ is spanned by the maps $\{r_{ij}\}$ and consequently is finite dimensional.
(2) If V is irreducible, W is irreducible, $V \not\cong W$ and $S(W)$ is defined analogously to $S(V)$ then $S(V) \perp S(W)$ as subspaces of $C^0(G, \mathbb{K})$
(3) If $R_g : C^0(G, \mathbb{K}) \to C^0(G, \mathbb{K})$ and $L : C^0(G,\mathbb{K}) \to C^0(G, \mathbb{K})$ are defined by

$$(R_g f)(x) = f(xg) \qquad (L_g f)(x) = f(g^{-1}x)$$

for $g, x \in G$, $f \in C^0(G, \mathbb{K})$ then $S(V)$ is both a R and L submodule of the G-module $C^0(G, \mathbb{K})$.

Proof
(1) Clearly s is linear over $\mathbb{K}$ since the group action is linear. Thus we see that $S(V)$ is spanned by $\{s(e_j^* \otimes e_i)\}$ indeed for $\alpha \in V^*$, $v \in V$

$$s(\alpha \otimes v)(g) = \alpha(gv) = \sum_j (\alpha^j e_j^*)(g \sum_i v^i e_i)$$

$$= \sum_j \sum_i \alpha^j v^i e_j^*(g e_i)$$

$$= \sum_j \sum_i \alpha^j v^i e_j^*\Big(\sum_k r_{ki}(g) e_k\Big)$$

$$= \sum_j \sum_i \alpha^j v^i r_{ji}(g) \qquad \forall g \in G$$

Thus

$$s(\alpha \otimes v) = \sum_j \sum_i \alpha^j v^i r_{ji} \in \langle r_{ji} \rangle$$

(1) follows.

(2) By (1) $S(V)$ is spanned by $\{r^V_{ij}\} \subseteq C^0(G,\mathbb{K})$ and $S(\overline{W})$ is spanned by $\{\overline{r^W_{kl}}\} \subseteq C^0(G,\mathbb{K})$.
But

$$\int_G r^V_{ij}(g)\,\overline{r^W_{kl}(g)}\,dg = 0$$

and so

$$\langle r^V_{ij}, r^W_{kl} \rangle = 0 \qquad \text{for all } i,j,k,l.$$

Thus $S(V) \perp S(\overline{W})$.

(3)
$$\begin{aligned} R_g(s(\alpha\otimes v))(x) &= s(\alpha\otimes v)(xg) \\ &= \alpha(xgv) \\ &= s(\alpha\otimes(gv))(x) \qquad \forall \alpha\in V^*, v\in V, g,x\in G \end{aligned}$$

$\therefore$ $R_g(s(\alpha\otimes v)) = s(\alpha\otimes(gv))$, i.e.
R_g acts on $s(\alpha\otimes v)$ by acting on v. Now

$$\begin{aligned} L_g(s(\alpha\otimes v))(x) &= s(\alpha\otimes v)(g^{-1}x) \\ &= \alpha(g^{-1}xv) \\ &= (g\cdot\alpha)(xv) \\ &= s((g\cdot\alpha)\otimes v)(x) \qquad \forall \alpha\in V^*, v\in V, g,x\in G \end{aligned}$$

So $L_g(s(\alpha\otimes v)) = s((g\cdot\alpha)\otimes v)$ acts by acting on α. The result follows easily.

<u>Definition</u> Let $R: G \to \mathrm{Aut}(C^0(G,\mathbb{K}))$ denote the R-representation defined above by

$$(R_g f)(x) = f(xg) \qquad \forall f\in C^0(G,\mathbb{K}),\ g,x\in G.$$

A function $f_0 \in C^0(G,\mathbb{K})$ is called a <u>representative function</u> for G iff

$$G \cdot_R f_0 = \{R_g f_0 \mid g\in G\}$$

is finite-dimensional. Notice that $G\cdot f_0$ is the smallest G-subspace of $C^0(G,\mathbb{K})$.

Theorem If $f_0 \in C^0(G, \mathbb{K})$ is a representative function for G then

$$G \cdot_L f_0 = \{ L_g f_0 \mid g \in G \}$$

is finite-dimensional. Moreover, the set of all representative functions is a $\mathbb{K}$-subalgebra of $C^0(G, \mathbb{K})$ which is closed under conjugation. We denote this subalgebra by $\mathcal{T}(G, \mathbb{K})$.

Proof Let $f_0 \in C^0(G, \mathbb{K})$ denote a representative function and let $V = G \cdot_R f_0$. Let $\{e_i\}$ denote a basis of V and $\{e_j^*\}$ the basis of V^* dual to $\{e_i\}$. Since $R_g f_0 \in V$,

$$R_g f_0 = \sum_j a_j(g) e_j \qquad \text{for } a_j(g) \in \mathbb{K}.$$

Now

$$\begin{aligned} s_V(e_j^* \otimes f_0)(g) &= e_j^*(R_g f) \\ &= e_j^*\Big(\sum_i a_i(g) e_i\Big) \\ &= a_j(g) \end{aligned}$$

Now

$$f_0(g) = f_0(1 \cdot g) = (R_g f)(1) = \sum_j a_j(g)\, e_j(1)$$

and

$$f_0(g) = \sum_j e_j(1)\, s_V(e_j^* \otimes f_0)(g) \qquad \forall g \in G$$

$$\therefore\ f_0 = \sum_j e_j(1)\, s_V(e_j^* \otimes f_0) \in S(V)$$

Now $S(V)$ is finite dimensional & is an L-subspace of $C^0(G, \mathbb{K})$. Since

$G \cdot_L f_0 \subseteq S(V)$, $G\, f_0$ is finite dimensional.

We now show that the set $\mathcal{J}(G, \mathbb{K})$ of representative functions is a $\mathbb{K}$-subalgebra of $C^0(G, \mathbb{K})$. Let $\varphi, \psi \in \mathcal{J}(G, \mathbb{K})$. Then $\varphi \in S(V)$, $\psi \in S(W)$ where $V = G \cdot_R \varphi$, $W = G \cdot_R \psi$. Now $\varphi + \psi \in S(V) + S(W)$ which is a finite dimensional G-submodule of $C^0(G, \mathbb{K})$, thus $G \cdot_R (\varphi + \psi)$ is finite dimensional and $\varphi + \psi \in \mathcal{J}(G, \mathbb{K})$. Also if $\lambda \in \mathbb{K}$ then $\lambda\varphi \in S(V)$ and $G \cdot_R (\lambda\varphi)$ is finite dimensional, thus $\lambda\varphi \in \mathcal{J}(G, \mathbb{K})$. Now for $g \in G$, $x \in G$

$$
\begin{aligned}
R_g(\varphi\psi)(x) &= (\varphi\psi)(xg) = \varphi(xg)\,\psi(xg) \\
&= (R_g\varphi)(x)\,(R_g\psi)(x)
\end{aligned}
$$

and

$$R_g(\varphi\psi) = (R_g\varphi)(R_g\psi) \in S(V) \cdot S(W)$$

But $R_g\varphi \in S(V) \Rightarrow R_g\varphi$ is a linear combination of $\{r^V_{ij}\}$ where r^V_{ij} are the components of the matrix representation induced by the R-action on V and the choice of a basis of V. Similarly $R_g\psi \in S(W) \Rightarrow R_g\psi$ is a linear combination of $\{r^W_{kl}\}$. So

$$R_g(\varphi\psi) = \sum \lambda^{ij} \mu^{kl}\, r^V_{ij}\, r^W_{kl}$$

and $R_g(\varphi\psi)$ is in the finite dimensional subspace spanned by $\{r^V_{ij}\, r^W_{kl}\}$ $\forall g \in G$. Thus

$G \cdot_R (\varphi\psi)$ is finite dimensional

and $\varphi\psi \in \mathcal{J}(G, \mathbb{K})$. It follows that $\mathcal{J}(G, \mathbb{K})$ is a subalgebra of $C^0(G, \mathbb{K})$.

To see that $\mathcal{J}(G, \mathbb{K})$ is closed under conjugation, let $\varphi \in \mathcal{J}(G, \mathbb{K})$, we show $\overline{\varphi} \in \mathcal{J}(G, \mathbb{K})$.
Since $\varphi \in \mathcal{J}(G, \mathbb{K})$,

$$V = \{ R_g \varphi \mid g \in G \}$$

is finite dimensional. Let $\{e_i\} \subseteq C^0(G, \mathbb{K})$ be a basis of V. Then $R_g \varphi = \sum_i a_i(g) e_i$
But

$$(R_g \overline{\varphi})(x) = \overline{\varphi}(xg) = \overline{\varphi(xg)} = \sum_i \overline{a_i(g)}\, \overline{e_i(x)}$$

Now $\{\overline{e_i}\}$ are linearly indept in $C^0(G, \mathbb{K})$ and we see that $R_g \overline{\varphi} = \sum_i \overline{a_i}(g) \overline{e_i}$ thus

$$\{ R_g \overline{\varphi} \mid g \in G \}$$

is contained in the span of $\{\overline{e_i}\}$ and so is finite dimensional. Thus $\overline{\varphi} \in \mathcal{J}(G, \mathbb{K})$.

Corollary If $f \in \mathcal{J}(G, \mathbb{K})$ then $f \in S(V)$ for some finite dimensional R-submodule of $C^0(G, \mathbb{K})$.

Proof $G \cdot_R f \subseteq S(V)$ where $V = G \cdot_R f$.

Let $\{V_\alpha\}$ denote all the elements of $\mathrm{Irr}(G, \mathbb{K})$ which are irreducible R-submodules of $C^0(G, \mathbb{K})$. For $\alpha \neq \beta$, $V_\alpha \not\cong V_\beta$ and $S(V_\alpha) \perp S(V_\beta)$
Observe that $f \in \mathcal{J}(G, \mathbb{K})$ implies that $f \in S(V)$ for some finite dimensional R-submodule V of $C^0(G, \mathbb{C})$. Now

$$V = \oplus_i n_{\alpha_i} V_{\alpha_i}$$

So $V = \bigoplus_i n_{\alpha_i} V_{\alpha_i} \subseteq \bigoplus_i n_{\alpha_i} S(V_{\alpha_i})$

Corollary $\mathcal{J}(G,\mathbb{K}) \subset \bigoplus_\alpha n_\alpha S(V_\alpha)$
$S(V_\alpha) \perp S(V_\beta)$

Proposition If U is an irreducible R-submodule of $C^0(G,\mathbb{K})$, then $S(U)$ is the direct sum of the distinct R-submodules $s_U(e_k^* \otimes_{\mathbb{K}} U)$ where $\{e_k\}$ is a basis of U and $\{e_k^*\}$ is the basis of U^* dual to $\{e_k\}$. Moreover $s_U(e_k^* \otimes_{\mathbb{K}} U)$ is G-isomorphic to U for each k. Thus $S(U)$ is the U isotypical part of $\mathcal{J}(G,\mathbb{K})$.

Proof Let $\{e_k\}$ be a basis of U + $\{e_k^*\}$ the dual basis.

$$U^* \otimes_{\mathbb{K}} U = \sum_k (e_k^* \otimes_{\mathbb{K}} U) \quad \text{(sum not nec. direct)}$$

Now

$$s_U(U^* \otimes_{\mathbb{K}} U) = \sum_k s_U(e_k^* \otimes U)$$

$$s_U(e_k^* \otimes U) = \{ s_U(e_k^* \otimes u) \mid u \in U \}$$

$$s_U(e_k^* \otimes u)(g) = e_k^*(gu) \qquad \forall g \in G,$$

$$\begin{aligned} R_g(s_U(e_k^* \otimes u))(x) &= s_U(e_k^* \otimes u)(xg) \\ &= e_k^*(xgu) \\ &= s_U[e_k^* \otimes gu](x) \qquad \forall x \in G \end{aligned}$$

So $R_g(s_U(e_k^* \otimes u)) = s_U(e_k^* \otimes (gu)) \quad \forall g \in G$

Thus $s_U(e_k^* \otimes u)$ is a morphism as a function of u, i.e.

$$u \longrightarrow s_U(e_k^* \otimes u)$$

is a morphism from U onto $s_U(e_k^* \otimes_{\mathbb{K}} U)$. The action R acts on $s_U(e_k^* \otimes u)$ on the u-term. We claim that this map is injective. If $s_U(e_k^* \otimes u) = 0$ then $e_k^*(gu) = 0$ $\forall g \in G$. But $G \cdot u$ is a submodule of U and so must either be 0 or all of U. Since $e_k^*(e_k) = 1$, e_k^* is not zero on all of U so $gu = 0$ $\forall g \in G$. Thus $u = 0$ and the mapping $u \longrightarrow s_U(e_k^* \otimes u)$ is an isomorphism of modules. Thus $s_U(e_k^* \otimes U)$ is irreducible $\forall k$. So for $k \neq l$ either $s_U(e_k^* \otimes U) \cap s_U(e_l^* \otimes U) = 0$ or else $s_U(e_k^* \otimes U) = s_U(e_l \otimes U)$. Thus

$$s_U(U^* \otimes_{\mathbb{K}} U) = \bigoplus_k s_U(e_k^* \otimes U)$$

But recall that $S(U)$ is the image of s_U so

$$S(U) = \bigoplus_k s_U(e_k^* \otimes_{\mathbb{K}} U).$$

Corollary If f is a representative function of G then there exist irreducible submodules E_i of $C^0(G, \mathbb{K})$ and $e_i \in E_i$ such that for some $\lambda_i \in \mathbb{K}$

$$f = \sum_i \lambda_i f_i(g e_i)$$

for some $f_i \in E_i^*$.

Thus $s_U(e_k^* \otimes u)$ is a morphism as a function of u, i.e.

$$u \longrightarrow s_U(e_k^* \otimes u)$$

is a morphism from U onto $s_U(e_k^* \otimes_{\mathbb{K}} U)$. The action R acts on $s_U(e_k^* \otimes u)$ on the u-term. We claim that this map is injective. If $s_U(e_k^* \otimes u) = 0$ then $e_k^*(gu) = 0$ $\forall g \in G$. But $G \cdot u$ is a submodule of U and so must either be 0 or all of U. Since $e_k^*(e_k) = 1$, e_k^* is not zero on all of U so $gu = 0$ $\forall g \in G$. Thus $u = 0$ and the mapping $u \to s_U(e_k^* \otimes u)$ is an isomorphism of modules. Thus $s_U(e_k^* \otimes U)$ is irreducible $\forall k$. So for $k \neq l$ either $s_U(e_k^* \otimes U) \cap s_U(e_l^* \otimes U) = 0$ or else $s_U(e_k^* \otimes U) = s_U(e_l \otimes U)$. Thus

$$s_U(U^* \otimes_{\mathbb{K}} U) = \bigoplus_k s_U(e_k^* \otimes U)$$

But recall that $S(U)$ is the image of s_U so

$S(U) = \bigoplus_k s_U(e_k^* \otimes_{\mathbb{K}} U)$.

Corollary If f is a representative function of G then there exist irreducible submodules E_i of $C^0(G, \mathbb{K})$ and $\hat{e}_i \in E_i$ such that

$$f = \sum_i f_i(g\hat{e}_i)$$

for some $f_i \in E_i^*$.

Proof It follows from the Theorem that if f is a representative function of G then there exists irreducible submodules E_i such that $f \in \sum_i S(E_i)$, thus $f = \sum_i h_i$ where $h_i \in S(E_i)$. Now $S(E_i) = \sum_k s_{E_i}(e_{ik}^* \otimes E_i)$ where $\{e_{ki}\}$ is a basis of E_i.

Now $S(E_i) = s_{E_i}(\langle e_{ik}^* \rangle \otimes E_i)$

so

$$h_i = s_{E_i}(f_i \otimes \hat{e}_i)$$

Where $f_i \in \langle e_{ik}^* \rangle \subseteq \mathrm{Hom}(E_i, K)$ and $\hat{e}_i \in E_i$ ($\hat{e}_i$ is a notation for an element of E_i, it is not necessarily part of a basis).

Thus

$$f(g) = \sum_i h_i(g) = \sum_i s_{E_i}(f_i \otimes \hat{e}_i)(g)$$

$$f(g) = \sum_i f_i(g\hat{e}_i).$$

The corollary follows.

Remark. This result is used in part of the proof of the Peter-Weyl theorem. In particular if f is a representative function one has for $x \in G$ and $l_x : E_i \to E_i$

$$\begin{aligned}\int_G f(gxg^{-1})\,dg &= \sum_i \int_G f_i(gxg^{-1}e_i)\,dg \\ &= \sum_i \int_G f_i(g\, l_x(g^{-1}e_i))\,dg \qquad (4.4 \text{ Page } 78) \\ &= \sum_i \left(\frac{1}{\dim E_i}\right) f_i(e_i)\, \chi_{E_i}(x) \qquad (\chi_{E_i}(x) = \mathrm{Tr}(l_x))\end{aligned}$$

Theorem Let G be a Lie group and $\mathfrak{g}$ its Lie algebra of left invariant vector fields. Let $\mathfrak{h}$ be a sub-Lie algebra of $\mathfrak{g}$. If $\varphi_1 : H_1 \to G$, $\varphi_2 : H_2 \to G$ are immersed Lie subgroups of G such that $d_e\varphi_i(T_eH) = \{X(e) \mid X \in \mathfrak{h}\}$ then there exists a Lie group isomorphism $\psi_0 : H_1 \to H_2$ such that $\varphi_1 = \varphi_2 \circ \psi_0$.

Proof Let $\mathcal{D}$ be the distribution on G defined by

$$\mathcal{D}(g) = \{ X(g) \mid X \in \mathfrak{h} \}$$

The proof of the theorem on page A30 shows that $\mathcal{D}$ is a smooth involutive distribution. Suppose $\varphi : H \to G$ is any immersed Lie subgroup of G such that $(d_e\varphi)(T_eH) = \mathfrak{h}$, more precisely assume

$$(T_e\varphi)(T_eH) = \{ X(e) \mid X \in \mathfrak{h} \}.$$

We claim $l_a \circ \varphi : H \to G$ is an integral manifold of $\mathcal{D}$ for each $a \in G$. Indeed,

$$d_e(l_a \circ \varphi)(T_eH) = dl_a(d_e\varphi(T_eH))$$

$$= dl_a(\{X(e) \mid X \in \mathfrak{h}\})$$

$$= \{X(a) \mid X \in \mathfrak{h}\}$$

$$= \mathcal{D}(a).$$

Now $\varphi(H)$ is a subgroup of G and $l_a(\varphi(H))$ is a coset for each $a \in G$. Thus the images of $l_a \circ \varphi$ and $l_b \circ \varphi$ are disjoint or are equal. Thus $l_a \circ \varphi : H \to G$ is a maximal integral

manifold for each $a \in G$. Thus if
$\varphi_1 : H_1 \to G$, $\varphi_2 : H_2 \to G$ are immersed
Lie subgroups of G then they both are
maximal integral manifolds of $\mathcal{D}$ through e and
by uniqueness of maximal integral manifolds
through a point such as e there exist
a diffeomorphism $\psi_0 : H_1 \to H_2$ such that

$$\begin{array}{ccc} H_1 & \xrightarrow{\psi_0} & H_2 \\ & {\scriptstyle\varphi_1}\searrow \quad \swarrow{\scriptstyle\varphi_2} & \\ & G & \end{array}$$

is commutative (see the proof on Page A29).
Now $\varphi_1 = \varphi_2 \circ \psi_0$ and since φ_1, φ_2 are
both injective it is easy to show that both
ψ_0 and its inverse are group homomorphisms:

$$\varphi_2(\psi_0(xy)) = \varphi_1(xy) = \varphi_1(x)\varphi_1(y) = \varphi_2(\psi_0(x))\varphi_2(\psi_0(y))$$
$$= \varphi_2(\psi_0(x)\psi_0(y))$$
$$\Rightarrow \psi_0(xy) = \psi_0(x)\psi_0(y)$$

with a similar calculation for ψ_0^{-1}.

<u>Theorem</u> Let G and H be Lie groups and
$\varphi : \mathfrak{g} \to \mathfrak{h}$ a Lie algebra homomorphism from
the Lie algebra $\mathfrak{g}$ of G into the Lie algebra $\mathfrak{h}$
of H. If $\Phi_1 : G \to H$, $\Phi_2 : G \to H$ are
Lie group homomorphisms such that $d_e\Phi_1 = \varphi = d_e\Phi_2$
then $\Phi_1 = \Phi_2$

Proof Given $\Phi_1, \Phi_2 : G \to H$ such that $d_e\Phi_1 = \varphi = d_e\Phi_2$ we define immersions

$$\hat{\Phi}_1 : G \to G\times H \qquad \hat{\Phi}_2 : G \to G\times H$$

of G into $G\times H$ by

$$\hat{\Phi}_i(g) = (g, \Phi_i(g)) \qquad g\in G, \ i=1,2.$$

Clearly $\hat{\Phi}_i$ is a group homomorphism for $i=1,2$ since Φ_i are homomorphisms. Also, for $g\in G$, $v \in T_gG$

$$(d_g\hat{\Phi}_i)(v) = (v, d_g\Phi_i(v)).$$

Thus $d_g\hat{\Phi}$ is injective. Since $\hat{\Phi}_i$ is injective $\hat{\Phi}_i : G \to G\times H$ is an immersion. Now

$$(d_e\hat{\Phi}_i)(v) = (v, d_e\Phi_i(v)) = (v, \varphi(v))$$

for all $v\in T_eG$ and for $i=1,2$. Thus the Lie algebra of G may be identified with T_eG which is mapped onto graph φ by $d_e\hat{\Phi}_i$ for $i=1,2$. Thus graph φ is a sub-Lie algebra of $\mathfrak{g}\times\mathfrak{h}$ and $\mathfrak{g}\times\mathfrak{h}$ is the Lie algebra of the Lie group $G\times H$; yet there are two immersed subgroups $\hat{\Phi}_i : G \to G\times H$ with Lie algebra graph φ. By the theorem on page A36 it follows that there exists a Lie group isomomorphism $\psi_0 : G \to G$ such that $\hat{\Phi}_1 = \hat{\Phi}_2 \circ \psi_0$. Thus

$$(g, \Phi_1(g)) = (\psi_0(g), \Phi_2(\psi_0(g)))$$

for all $g\in G$ and ψ_0 = identity mapping and

$$\Phi_1 = \Phi_2 \circ \psi_0 = \Phi_2$$