HW 1

With the maps $\varphi : \mathbb{C} \to gl(2, \mathbb{R})$, $\psi : gl(n, \mathbb{C}) \to gl(2n, \mathbb{R})$

$$\varphi(z) = \begin{pmatrix} \operatorname{Re} z & -\operatorname{Im} z \\ \operatorname{Im} z & \operatorname{Re} z \end{pmatrix}$$

$$\psi(A) = \begin{pmatrix} \varphi(A_1^1) & \cdots & \varphi(A_n^1) \\ \vdots & \ddots & \vdots \\ \varphi(A_1^n) & \cdots & \varphi(A_n^n) \end{pmatrix}$$

worksheet 1

Since $\mathbb{C}$ is a field for $z, w \in \mathbb{C}$, $\lambda \in \mathbb{R}$

$z + \lambda w \in \mathbb{C}$ and

$$\varphi(z+\lambda w) = \begin{pmatrix} \operatorname{Re}(z+\lambda w) & -\operatorname{Im}(z+\lambda w) \\ \operatorname{Im}(z+\lambda w) & \operatorname{Re}(z+\lambda w) \end{pmatrix}$$

$$= \begin{pmatrix} \operatorname{Re}(z) + \lambda \operatorname{Re}(w) & -(\operatorname{Im}(z) - \lambda \operatorname{Im}(w)) \\ \operatorname{Im}(z) + \lambda \operatorname{Im}(w) & \operatorname{Re}(z) + \lambda \operatorname{Re}(w) \end{pmatrix}$$

$$= \begin{pmatrix} \operatorname{Re}(z) & -\operatorname{Im}(z) \\ \operatorname{Im}(z) & \operatorname{Re}(z) \end{pmatrix} + \lambda \begin{pmatrix} \operatorname{Re}(w) & -\operatorname{Im}(w) \\ \operatorname{Im}(w) & \operatorname{Re}(w) \end{pmatrix}$$

$$= \varphi(z) + \lambda \varphi(w)$$

since addition in $\mathbb{C}$ is

$z + w = (\operatorname{Re} z + i \operatorname{Im} z) + (\operatorname{Re} w + i \operatorname{Im} w) =$
$(\operatorname{Re} z + \operatorname{Re} w) + i(\operatorname{Im} z + \operatorname{Im} w) = \operatorname{Re}(z+w) + i \operatorname{Im}(z+w)$

also $zw \in \mathbb{C}$

$$\varphi(zw) = \begin{pmatrix} Re(zw) & -Im(zw) \\ Im(zw) & Re(zw) \end{pmatrix}$$

$$= \begin{pmatrix} Re z\, Re w - Im z\, Im w & -(Im z\, Re w + Im w\, Re z) \\ Im z\, Re w + Im w\, Re z & Re z\, Re w - Im z\, Im w \end{pmatrix}$$

$$= \begin{pmatrix} Re(z) & -Im(z) \\ Im(z) & Re(z) \end{pmatrix} \cdot \begin{pmatrix} Re(w) & -Im(w) \\ Im(w) & Re(w) \end{pmatrix}$$

$$= \varphi(z)\,\varphi(w)$$

Since

$$z \cdot w = (Re z + i\, Im z)(Re w + i\, Im w)$$
$$= Re z\, Re w - Im z\, Im w + i(Im z\, Re w + Im w\, Re z)$$
$$= Re(zw) + i\, Im(zw)$$

②

Worksheet #2

Since $gl(n, \mathbb{C})$ is a ring $A + \lambda B \in gl(n, \mathbb{C})$ for $A, B \in gl(n, \mathbb{C})$ and $\lambda \in \mathbb{R}$

and by definition $(A+\lambda B)^i_j = A^i_j + \lambda B^i_j$

then
$$\Psi(A+\lambda B) = \begin{pmatrix} \varphi[(A+\lambda B)^1_1] & \cdots & \varphi[(A+\lambda B)^1_n] \\ \vdots & \ddots & \vdots \\ \varphi[(A+\lambda B)^n_1] & \cdots & \varphi[(A+\lambda B)^n_n] \end{pmatrix}$$

by ①

$$= \begin{pmatrix} \varphi(A^1_1) + \lambda\varphi(B^1_1) & \cdots & \varphi(A^1_n) + \lambda\varphi(B^1_n) \\ \vdots & \ddots & \vdots \\ \varphi(A^n_1) + \lambda\varphi(B^n_1) & \cdots & \varphi(A^n_n) + \lambda\varphi(B^n_n) \end{pmatrix}$$

by matrix addition

$$= \begin{pmatrix} \varphi(A^1_1) & \cdots & \varphi(A^1_n) \\ \vdots & \ddots & \vdots \\ \varphi(A^n_1) & \cdots & \varphi(A^n_n) \end{pmatrix} + \lambda \begin{pmatrix} \varphi(B^1_1) & \cdots & \varphi(B^1_n) \\ \vdots & \ddots & \vdots \\ \varphi(B^n_1) & \cdots & \varphi(B^n_n) \end{pmatrix}$$

$$= \Psi(A) + \lambda\Psi(B)$$

Now for $A, B \in gl(n, \mathbb{C})$ then $AB \in gl(n, \mathbb{C})$ and by definition $(AB)^i_j = A^i_k B^k_j$

$$\Psi(AB) = \begin{pmatrix} \varphi((AB)^1_1) & \cdots & \varphi((AB)^1_n) \\ \vdots & & \vdots \\ \varphi((AB)^n_1) & \cdots & \varphi((AB)^n_n) \end{pmatrix}$$

$$= \begin{pmatrix} \varphi(A^1_k B^k_1) & \cdots & \varphi(A^1_k B^k_n) \\ \vdots & & \vdots \\ \varphi(A^n_k B^k_1) & & \varphi(A^n_k B^k_n) \end{pmatrix}$$

since φ is linear and a homomorphism

Second, might be useful here!

$$= \begin{pmatrix} \sum \varphi(A^1_k)\varphi(B^k_1) & \cdots & \sum \varphi(A^1_k)\varphi(B^k_n) \\ \vdots & & \vdots \\ \sum \varphi(A^n_k)\varphi(B^k_1) & \cdots & \sum \varphi(A^n_k)\varphi(B^k_n) \end{pmatrix}$$

Since $\varphi(A^i_k)\varphi(B^k_j) = \varphi(A^i_1)\varphi(B^1_j) + \varphi(A^i_2)\varphi(B^2_j) + \cdots + \varphi(A^i_n)\varphi(B^n_j)$

$$= \begin{pmatrix} \varphi(A^1_1) & \cdots & \varphi(A^1_n) \\ \vdots & \ddots & \vdots \\ \varphi(A^n_1) & \cdots & \varphi(A^n_n) \end{pmatrix} \cdot \begin{pmatrix} \varphi(B^1_1) & \cdots & \varphi(B^1_n) \\ \vdots & \ddots & \vdots \\ \varphi(B^n_1) & \cdots & \varphi(B^n_n) \end{pmatrix}$$

$$= \Psi(A)\Psi(B)$$

Worksheet #3

We show that Ψ is injective and continuous.

Suppose $\Psi(A) = \Psi(B)$ then

$$\begin{pmatrix} \varphi(A^1_1) & \cdots & \varphi(A^1_n) \\ \vdots & & \\ \varphi(A^n_1) & & \varphi(A^n_n) \end{pmatrix} = \begin{pmatrix} \varphi(B^1_1) & \cdots & \varphi(B^1_n) \\ \vdots & & \vdots \\ \varphi(B^n_1) & \cdots & \varphi(B^n_n) \end{pmatrix}$$

this implies $\varphi(A^i_j) = \varphi(B^i_j) \quad \forall i, j.$

$$\Rightarrow \begin{pmatrix} Re(A^i_j) & -Im(A^i_j) \\ Im(A^i_j) & Re(A^i_j) \end{pmatrix} = \begin{pmatrix} Re(B^i_j) & -Im(B^i_j) \\ Im(B^i_j) & Re(B^i_j) \end{pmatrix}$$

hence $Re(A^i_j) = Re(B^i_j)$
$Im(A^i_j) = Im(B^i_j)$

therefore $A^i_j = B^i_j \quad \forall i, j$

Since this is true $\forall i, j$

$$A = B$$

Continued on page 4

worksheet #4

Let $A \in gl(n, \mathbb{C})$

From the matrix of $A_{\mathbb{R}}$ take the rows labeled by e_j and ie_k

then $e_j \cdot ie_k = [\operatorname{Re} A_j^1, \operatorname{Im} A_j^1, \operatorname{Re} A_j^2, \operatorname{Im} A_j^2, \ldots, \operatorname{Re} A_j^n, \operatorname{Im} A_j^n] \begin{bmatrix} -\operatorname{Im} A_k^1 \\ \operatorname{Re} A_k^1 \\ \vdots \\ -\operatorname{Im} A_k^n \\ \operatorname{Re} A_k^n \end{bmatrix}$

$= -\operatorname{Re} A_j^1 \operatorname{Im} A_k^1 + \operatorname{Im} A_j^1 \operatorname{Re} A_k^1 + \ldots + -\operatorname{Re} A_j^n \operatorname{Im} A_k^n + \operatorname{Im} A_j^n \operatorname{Re} A_k^n$

$= (\operatorname{Im} A_j^1 \operatorname{Re} A_k^1 - \operatorname{Re} A_j^1 \operatorname{Im} A_k^1) + \ldots + (\operatorname{Im} A_j^n \operatorname{Re} A_k^n - \operatorname{Re} A_j^n \operatorname{Im} A_k^n)$

notice that the grouped terms equal 0 if $j = k$

therefore $e_j \cdot ie_j = 0$

where e_j denotes the row labeled by e_j in $A_{\mathbb{R}}$

" ie_j " " " ie_j "

Dr. Fulp we didn't use #4 in any problem but we did use this lemma

lemma $\varphi(\overline{x}) = \varphi(x)^t$

proof

let $x \in \mathbb{C}$ then $x = \operatorname{Re}(x) + i \operatorname{Im}(x)$ and $\overline{x} = \operatorname{Re}(x) - i \operatorname{Im}(x)$

$$\varphi(\overline{x}) = \begin{pmatrix} \operatorname{Re}(x) & -(-\operatorname{Im}(x)) \\ -\operatorname{Im}(x) & \operatorname{Re}(x) \end{pmatrix} = \begin{pmatrix} \operatorname{Re}(x) & \operatorname{Im}(x) \\ -\operatorname{Im}(x) & \operatorname{Re}(x) \end{pmatrix}$$

$$= \begin{pmatrix} \operatorname{Re}(x) & -\operatorname{Im}(x) \\ \operatorname{Im}(x) & \operatorname{Re}(x) \end{pmatrix}^t = \varphi(x)^t$$

To show Ψ is continuous we will need the following fact

$$\|\varphi(x)\|_{\mathbb{R}}^2 = 2\|x\|_{\mathbb{C}}^2$$

proof

Let $x \in \mathbb{C}$

$$\|\varphi(x)\|^2 = (\varphi(x)_1^1)^2 + (\varphi(x)_2^1)^2 + (\varphi(x)_1^2)^2 + (\varphi(x)_2^2)^2$$
$$= (Re(x))^2 + (-Im(x))^2 + (Im(x))^2 + (Re(x))^2$$
$$= 2(Re(x))^2 + 2(Im(x))^2$$
$$= 2\|x\|^2$$

Then $\|\Psi(A)\|^2 = \sum_{ij}^{n} \|\varphi(A_j^i)\|^2$ [Here we have regrouped the terms since $\Psi(A)$ is a block matrix]

$$= \sum_{ij}^{n} 2\|A_j^i\|^2$$
$$= 2\sum_{ij}^{n} \|A_j^i\|^2$$
$$= 2\|A\|^2$$

and since $\|\,\| \geq 0$ $\quad \|\Psi(A)\| = \sqrt{2}\,\|A\|$

Given $\varepsilon > 0$, let $A, B \in gl(n, \mathbb{C})$ such that

$$\|A - B\| \leq \frac{\varepsilon}{2\sqrt{2}}$$

then $\|\Psi(A) - \Psi(B)\| = \|\Psi(A-B)\| = \sqrt{2}\,\|A - B\|$

$$\leq \sqrt{2}\,\frac{\varepsilon}{2\sqrt{2}} = \frac{\varepsilon}{2} < \varepsilon$$

therefore Ψ is continuous.