HW 2

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48/50

Problem #8 Let $B = \{$ group of triangular complex matrices with positive real diagonals

then define $\phi : U(n) \times B \to Gl(n, \mathbb{C})$

by $\varphi(C, A) = C \cdot A$

$$\det(CA) = \det(C)\det(A) = \text{trace}(A)\det(C) \neq 0$$

Since A is triangular with pos. diagonal

Since A has real <u>positive</u> diagonal and

$U(n) \subseteq Gl(n, \mathbb{C}) \Rightarrow \det(C) \neq 0$

therefore φ is well defined.

Let $M \in Gl(n, \mathbb{C})$ then by def $\det(M) \neq 0$

so M has full rank.

Since M has full rank we can use Gram-Schmidt (G-S) to QR decompose M

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write $M = [m_1 | m_2 | m_3 | \cdots | m_n]$

then by the method of G-S we find $[e_1 | e_2 | \cdots | e_n]$ where $\langle e_i, e_j \rangle = 0$ $i \neq j$

$$u_K = m_K - \sum_{j=1}^{K-1} P(e_j) m_K \quad \text{where} \quad e_K = \frac{u_K}{\|u_K\|}$$

$u_1 = m_1$

where $P(u)V$ is the projection $P(u)V = \dfrac{\langle V, u \rangle}{\langle u, u \rangle} u$

we can solve for m_j

$$m_1 = u_1$$

$$\vdots$$

$$m_k = \sum_{j=1}^{k-1} P(e_j) m_k + e_k \|u_k\| = \sum_{j=1}^{k-1} \frac{\langle e_j, m_k \rangle}{\langle e_k, e_k \rangle} e_k + e_k \|u_k\|$$

Since $\langle e_n, e_n \rangle = 1$

$$= \sum_{j=1}^{k-1} \langle e_j, m_k \rangle e_k + e_k \|u_k\|$$

then we can write

$$[m_1 | m_2 | \cdots | m_n] = [e_1 | \cdots | e_n] \begin{pmatrix} \|u_1\| & \langle e_1, m_2 \rangle & \langle e_1, m_3 \rangle & \cdots \\ & \|u_2\| & \langle e_2, m_3 \rangle & \cdots \\ & & \|u_3\| & \\ \bigcirc & & & \ddots \\ & & & & \|u_n\| \end{pmatrix}$$

$$= [e_1 | \cdots | e_n] R = QR$$

notice that $R \in B$ ie, triangular with positive diagonal

and for
$Q = [e_1 | \cdots | e_n]$

then $$Q^\dagger Q = [e_1 | \cdots | e_n]^\dagger [e_1 | \cdots | e_n] = \begin{bmatrix} \bar{e}_1^T \\ \bar{e}_2^T \\ \bar{e}_3^T \\ \vdots \\ \bar{e}_n^T \end{bmatrix} [e_1 | \cdots | e_n]$$

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$$\text{Since} = \begin{pmatrix} \bar{e}_1^T e_1 & \bar{e}_1^T e_2 & \cdots & \bar{e}_1^T e_n \\ \vdots & & \ddots & \\ \bar{e}_n^T e_1 & \cdots & & \bar{e}_n^T e_n \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix} = I$$

$$\boxed{e_k = \begin{bmatrix} e_k^1 \\ e_k^2 \\ \vdots \\ e_k^n \end{bmatrix}, \; e_k^i \in \mathbb{C}}$$

notation

Since

$$\bar{e}_i^T e_j = \sum_k \bar{e}_i^k e_j^k = \langle e_i, e_j \rangle$$

✓

and $\langle e_i, e_i \rangle = 1$

$\langle e_i, e_j \rangle = 0$ by construction

therefore $\underline{Q^\dagger Q = I(n)}$ ← implies $|\det Q| = 1$

now notice

$$\det(M) = \det(QR) = \det(Q)\det(R)$$

$$= \det(Q)\left[\sum_{k=1}^{n} \|u_k\|\right]$$

OK

$$\Rightarrow \underline{\det(Q)} = \frac{\det(M)}{\sum_{k=1}^{n} \|u_k\|} \underline{\neq 0}$$

since $\det(M) \neq 0$ and

$\sum_{k=1}^{n} \|u_k\| > 0$ since $\|u_1\| = \|M_1\| \neq 0$

Since $Q^\dagger Q = I$ and $\det(Q) \neq 0$

$\Rightarrow$

$Q \in U(n)$

Note

$U(n) = \{A \mid A^\dagger A = I\}$

implies $(\overline{\det A})\det A = 1$

Define $\Psi: Gl(n,\mathbb{C}) \to U(n)\times B$

as $\Psi(M) = (Q,R)$ where $M = QR$ after Gram-Schmidt

Need to show this is well-defined
Can you write $M = Q_1R_1$ and $M = Q_2R_2$ for different Q_1, Q_2 & R_1, R_2?

Since G-S QR decomposition is unique

then $\varphi(\Psi(M)) = \varphi(Q,R) = \underline{QR} = M$

and $\Psi(\varphi(C,A)) = \Psi(CA) = \underline{CA}$

Really need to show φ injective to get Ψ well defined and to see that $\varphi^{-1} = \Psi$

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therefore $\Psi = \varphi^{-1}$. Both φ and $\Psi = \varphi^{-1}$ <u>are smooth</u> since φ is bilinear and $\Psi = \varphi^{-1}$ involves polynomials of the coeff of a matrix from $Gl(n,\mathbb{C})$

Since φ is smooth and φ^{-1} exists and is smooth then

$$U(n)\times B \underset{\text{diffeo}}{\cong} Gl(n,\mathbb{C})$$ as manifolds.

Finally, also consider $\rho: B \to \mathbb{R}_+^n \times \mathbb{R}^{n^2-n}$

$$\rho(A) = \rho\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \ddots & & \vdots \\ 0 & & & a_{nn} \end{pmatrix} = (a_{11}, a_{22}, \ldots a_{nn}, Re(a_{12}), R(a_{13}), \ldots, Re(a_{n-1,n}), Im(a_{12}), \ldots Im(a_{n-1,n}))$$

✓ ρ is a linear map so it is smooth with smooth inverse therefore

$$\underline{Gl(n,\mathbb{C})} \underset{\varphi^{-1}}{\cong} U(n)\times B \underset{\rho}{\cong} U(n)\times\mathbb{R}_+^n\times\mathbb{R}^{n^2-n} \underset{\log}{\cong} \underline{U(n)\times\mathbb{R}^{n^2}}$$

PROBLEM 12 Show $O(2n+1) \cong SO(2n+1) \times \mathbb{Z}_2$ direct product

Define $\varphi : O(2n+1) \longrightarrow SO(2n+1) \times \{-1,1\}$ as follows,

$$\varphi(A) \equiv \left(\frac{A}{\det A} , \det A \right) \quad \text{for } A \in O(2n+1)$$

to begin we verify the codomain is accurately labeled, note $A \in O(2n+1) \Leftrightarrow A^T A = I \Rightarrow \det(A^T A) = \det(I)$

$$\Rightarrow \det(A^T)\det(A) = 1$$
$$\Rightarrow (\det(A))^2 = 1$$
$$\Rightarrow \det(A) = \pm 1$$

thus $\det(A) \in \{-1,1\}$ as needed. Also

$$\det\left(\frac{A}{\det A}\right) = \frac{1}{\det A}\det(A) = 1 \quad \therefore \quad \frac{A}{\det A} \in SO(2n+1).$$

Hence $\varphi(O(2n+1)) \subset SO(2n+1) \times \mathbb{Z}_2$ using $\mathbb{Z}_2 = \{-1,1\}$ multiplicatively.

φ a homomorphism?

$$\varphi(AB) = \left(\frac{AB}{\det(AB)} , \det(AB) \right)$$
$$= \left(\frac{A}{\det(A)} \frac{B}{\det(B)} , \det(A)\det(B) \right)$$
$$= \left(\frac{A}{\det(A)} , \det(A) \right) \cdot \left(\frac{B}{\det(B)} , \det(B) \right)$$
$$= \varphi(A)\varphi(B).$$

group multiplication of the direct product.

Where we have used that if G & H are groups then $G \times H$ the direct product has multiplication defined component-wise $(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1 h_2)$ $\forall g_1, g_2 \in G$ & $\forall h_1, h_2 \in H$.

PROBLEM 12 continued

Is φ invertible ?

Claim $\varphi^{-1}(X,Y) = XY \quad \forall\, (X,Y)\in SO(2n+1)\times\{-1,1\}$.

Observe, if $A\in O(2n+1)$,

$$\begin{aligned}\varphi^{-1}(\varphi(A)) &= \varphi^{-1}\left(\frac{A}{\det(A)},\ \det(A)\right)\\ &= \frac{A}{\cancel{\det(A)}}\cancel{\det(A)} \qquad (\det(A)=\pm1\neq 0)\\ &= A\end{aligned}$$

If $B\in SO(2n+1)$, $Z\in\{-1,1\}$ then,

$$\begin{aligned}\varphi(\varphi^{-1}(B,Z)) &= \varphi(BZ)\\ &= \left(\frac{BZ}{\det(BZ)},\ \det(BZ)\right)\\ &= \left(\frac{Z}{Z}\,\frac{B}{\underset{\uparrow \text{since } 2n+1 \text{ is odd!}}{\det B}},\ Z\det(B)\right)\\ &= (B,\ Z) \qquad \text{since } Z\neq 0 \ \&\ \det(B)=1\end{aligned}$$

✓

Thus $\varphi^{-1} : SO(2n+1)\times\{-1,1\}\to O(2n+1)$ is truly the inverse of the mapping φ. Thus φ is a group isomorphism. Moreover since φ & φ^{-1} are both defined in terms of rational functions (with non-zero denominators) of the entries of the matrices involved we find that φ and φ^{-1} are smooth mappings relative to the usual chart structure inherited* from $gl(2n+1,\mathbb{R})$ thus the following is a Lie Group Isomorphism

OK

$$O(2n+1)\cong SO(2n+1)\times\mathbb{Z}_2 \longleftarrow \text{Isomorphism}$$

yes

* I suppose there would be two charts one for each connected component.

$O(2n) \cong So(2n) \times_\tau \{I, \eta\}$ where $\tau(z)(A) = zAz$ and $(g_1, h_1), (g_2, h_2) \in So(2n) \times_\tau \{I, \eta\}$ are multiplied according to the semi-direct product,

$$(g_1, h_1) \cdot (g_2, h_2) \equiv (g_1\, \tau(h_1)(g_2), h_1 h_2)$$

Define: $\varphi : O(2n) \longrightarrow So(2n) \times_\tau \{I, \eta\}$,

$$\varphi(B) = \begin{cases} (B, I) & B \in So(2n) \\ (B\eta, \eta) & B \in So(2n)\eta \end{cases} \qquad \Bigg| \quad \eta = \begin{pmatrix} \overset{1}{1} & & & \\ & \overbrace{-1 & & \\ & & -1 & \\ & & & \ddots \\ & & & & -1}^{2n-1} \end{pmatrix}$$

for each $B \in O(2n)$. We show $O(2n) = So(2n) \sqcup So(2n)\eta$

disjoint union.

let $A \in O(2n)$ then for the same reasons as in the odd-dimensional case $\det(A) = \pm 1$.

I. CASE: $\det(A) = 1$

$$A \in So(2n) \subset So(2n) \sqcup So(2n)\eta \qquad \therefore\ O(2n) \subset So(2n) \sqcup So(2n)\eta$$

II. CASE: $\det(A) = -1$

$$A = A\eta\eta = (A\eta)\eta$$

noting $\det(A\eta) = \det(A)\det(\eta) = (-1)(-1) = 1$ we see that $(A\eta) \in So(2n)$ hence $(A\eta)\eta \in So(2n)\eta$.

thus $A \in So(2n)\eta \subset So(2n) \sqcup So(2n)\eta$.

. the reverse inclusions are trivial

$$\begin{cases} A \in So(2n) \Rightarrow A \in O(2n) \\ A\eta \in So(2n)\eta \Rightarrow (A\eta)^T(A\eta) = \eta^T A^T A \eta = \eta^T I \eta = \eta^T\eta = I. \end{cases}$$

(A ∈ so(2n) again)

Nice

$$\longrightarrow So(2n) \sqcup So(2n)\eta \subset O(2n)$$

$$\therefore\ \underline{O(2n) = So(2n) \sqcup So(2n)\eta}$$

PROBLEM 12 continued

last page we verified that the domain of φ was really $O(2n)$ now we verify $\varphi(O(2n)) \subset SO(2n) \times_\tau \{I, \eta\}$,

If $B \in SO(2n)$ then $(B, I) \in SO(2n) \times_\tau \{I, \eta\}$ clearly.

If $B = A\eta \in SO(2n)\eta$ where $A \in SO(2n)$ then since $\eta^2 = I$,

$$(B\eta)^T B\eta = \eta^T B^T B \eta = \eta^T I \eta = \eta\eta = I.$$

$$\det(B\eta) = \det(A\eta\eta) = \det(A) = 1$$

Hence $(B\eta, \eta) \in SO(2n) \times_\tau \{I, \eta\}$. Thus the domain & codomain of φ are well-defined.

Claim: φ is homomorphism of groups

We need to show $\varphi(AB) = \varphi(A)\varphi(B)$ for 4-cases since $A, B \in O(2n) = SO(2n) \sqcup SO(2n)\eta$.

Case I: $A, B \in SO(2n)$

Notice $\det(AB) = \det(A)\det(B) = 1$, $AB \in SO(2n)$.

$$\varphi(AB) = (AB, I)$$

$$\begin{aligned} \varphi(A)\varphi(B) &= (A, I) \cdot (B, I) \\ &= (A\,\tau(I)(B), II) \\ &= (AIBI, I) \\ &= (AB, I) = \varphi(AB) \end{aligned}$$

PROBLEM 12 continued

CASE II. $A \in SO(2n)$, $B \in SO(2n)\eta$.

Notice $B = C\eta$ for some $C \in SO(2n)$ thus $AB = AC\eta \in SO(2n)\eta$ as clearly $AC \in SO(2n)$.

$$\varphi(AB) = (AB\eta, \eta)$$

$$\begin{aligned} \varphi(A)\varphi(B) &= (A, I)\cdot(B\eta, \eta) \\ &= (A\tau(I)(B\eta), I\eta) \quad \leftarrow \text{semi-direct product} \\ &= (AIB\eta I, \eta) \\ &= (AB\eta, \eta) = \varphi(AB). \end{aligned}$$

CASE III. $B \in SO(2n)\eta$, $A \in SO(2n)$

Note $BA \in SO(2n)\eta$ since $\det(BA) = \det(B)\det(A) = -1$ and $O(2n) = SO(2n) \sqcup SO(2n)\eta$ if $BA \notin SO(2n)$ it must lie inside the other half, $BA \in SO(2n)\eta$.

$$\varphi(BA) = (BA\eta, \eta)$$

$$\begin{aligned} \varphi(B)\varphi(A) &= (B\eta, \eta)\cdot(A, I) \\ &= (B\eta\,\tau(\eta)(A), \eta I) \quad \leftarrow \text{semi-direct product} \\ &= (B\eta\,\eta A\eta, \eta) \\ &= (BA\eta, \eta) = \varphi(BA). \end{aligned}$$

CASE IV : $A, B \in SO(2n)\eta)$

$\exists\, C, D \in SO(2n)$ so that $A = C\eta$ & $B = D\eta$ thus

$$\begin{aligned} \det(AB) &= \det(C\eta\, D\eta) \\ &= \det(C)\det(\eta)\det(D)\det(\eta) \\ &= \det(\eta)^2 \\ &= 1 \quad \therefore \quad AB \in SO(2n). \end{aligned}$$

$$\varphi(AB) = (AB, I)$$

$$\begin{aligned} \varphi(A)\varphi(B) &= (A\eta, \eta)\cdot(B\eta, \eta) \\ &= (A\eta\ \tau(\eta)(B\eta), \eta\eta) \\ &= (A\eta\eta B\eta\eta, I) \\ &= (AIBI, I) \\ &= (AB, I) = \varphi(AB). \end{aligned}$$

Therefore, $\varphi(AB) = \varphi(A)\varphi(B) \quad \forall A, B \in O(2n).$

Claim: $\varphi^{-1} : SO(2n) \times_\tau \{I, \eta\}$ can be defined as in the last odd-dimensional case,

$$\varphi^{-1}((X, Y)) \equiv XY.$$

PROBLEM 12 continued

Verify that φ^{-1} is the inverse of φ as claimed, $B \in O(2n)$

$$\varphi^{-1}(\varphi(B)) = \varphi^{-1}\left(\begin{cases} (B, I) & : B \in SO(2n) \\ (B\eta, \eta) & : B \in SO(2n)\eta \end{cases} \right)$$

$$= \begin{cases} BI & : B \in SO(2n) \\ B\eta\eta & : B \in SO(2n)\eta \end{cases}$$

$$= B \qquad \text{since} \quad \eta^2 = I.$$

Let $(X, Y) \in SO(2n) \times_\tau \{I, \eta\}$, $X \in SO(2n)$ and $Y = I$ or η,

$$\varphi(\varphi^{-1}(X, Y)) = \varphi(XY)$$

$$= \begin{cases} (XY, I) & , XY \in SO(2n) & Ⓘ \\ (XY\eta, \eta) & , XY \in SO(2n)\eta & Ⓘⓘ \end{cases}$$

Break up into cases.

Ⓘ. $X \in SO(2n)$ & $XY \in SO(2n)$

$1 = \det(XY) = \underline{\det(X)\det(Y)} \Rightarrow \det(Y) = 1 \therefore \underline{Y = I}$.

$\Rightarrow (XY, I) = (X, I) = (X, Y)$.

Ⓘⓘ. $X \in SO(2n)$ & $XY \in SO(2n)\eta$

$-1 = \det(XY) = \det(X)\det(Y) \Rightarrow \det(Y) = -1 \therefore \underline{Y = \eta}$

✓ $\Rightarrow (XY\eta, \eta) = (X\eta\eta, \eta) = (X, \eta) = (X, Y)$.

Thus $\varphi(\varphi^{-1}(X, Y)) = (X, Y) \quad \forall (X, Y) \in SO(2n) \times_\tau \{I, \eta\}$.

$\therefore \varphi$ is a isomorphism & $O(2n) \cong SO(2n) \times_\tau \mathbb{Z}_2$ where $\mathbb{Z}_2 \cong \{I, \eta\}$. It is smooth and φ^{-1} is smooth for same reasons (as in the previous case.

OK but not quite complete