LAST TEST PROBLEM (MIDTERM #(5)) JAMES COOK

- Show kernel (P) is discrete

$\ker(P) \equiv \{ [\alpha] \in \tilde{G} \mid P([\alpha]) = e \}$

we argued P was group homomorphism so $e_{\tilde{G}} \equiv [e]$ is clearly in $\ker(P)$, $P([e]) = e(1) = e \in G$.

Let (U, x) be a chart about e in G then we saw $(\tilde{V}, y_{\tilde{V}} = x \circ P|_{\tilde{V}})$ provides a chart on $\tilde{G}$. However, $P^{-1}(U)$ is covered by more than just on $\tilde{V}$ in general since P^{-1} may be "multiply-valued". However since $\tilde{G}$ is Hausdorff (we proved earlier) we can write

$$P^{-1}(U) = \bigcup_{\alpha} \tilde{V}_\alpha \quad \text{with } \tilde{V}_\alpha \cap \tilde{V}_\beta = \phi \text{ for } \alpha \neq \beta.$$

such that $P(\tilde{V}_\alpha) = U$ and $P|_{\tilde{V}_\alpha} : \tilde{V}_\alpha \to U$ is a homeomorphism. Thus $P|_{\tilde{V}_\alpha}$ is injective on $\tilde{V}_\alpha$.

Let $[\alpha_0] \in \ker(P)$ then $\exists \tilde{V}_0$ with $[\alpha_0] \in \tilde{V}_0$ and $P|_{\tilde{V}_0} : \tilde{V}_0 \to U$ injective. Let $[\beta] \in \tilde{V}_0$ and suppose $P([\beta]) = e$ as well. Then

$$P([\alpha_0]) = e = P([\beta])$$

$$\Rightarrow [\alpha_0] = [\beta] \quad \Rightarrow \quad \ker P \cap \tilde{V}_0 = \{[\alpha_0]\}$$

Thus $[\alpha_0]$ is isolated by $\tilde{V}_0$ which is an open set in $\tilde{G}$. But $[\alpha_0]$ is arbitrary thus each point in $\ker P$ is isolated $\therefore$ <u>$\ker P$ is discrete</u>

Take Home Part of Final Exam

(1) Consider a manifold $\mathcal{M}$ with a maximal atlas $\mathcal{A}_{\mathcal{M}}$ and a nonvanishing n-form ω. We seek to show $\exists\, \mathcal{A}_{\mathcal{M}}^{+}$ a subatlas of $\mathcal{A}_{\mathcal{M}}$ such that $\det\left[J_{y\circ x^{-1}}(x(q)) \right] > 0$ for $(U,x), (V,y) \in \mathcal{A}_{\mathcal{M}}^{+}$ where $U \cap V \neq \phi$ and $q \in U \cap V$.

$$\boxed{\text{Def}^{n}/\ \mathcal{A}_{\mathcal{M}}^{+} = \left\{ (U,x) \in \mathcal{A}_{\mathcal{M}} \;\middle|\; \omega\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\right) > 0 \right\}}$$

Notice that we may write ω in terms of a function f and the differentials dx^i (locally that is)

$$\omega = f\, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$$

we argue $f(p) > 0 \ \forall p \in U$,

$$\omega\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\right) = f(dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n)\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\right)$$

$$= f \sum_{\sigma} \operatorname{sgn}(\sigma)\, dx_{i_{\sigma(1)}} \otimes \cdots \otimes dx_{i_{\sigma(n)}}\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$$

$$= f \sum_{\sigma} \operatorname{sgn}(\sigma)\, \underbrace{dx_{i_{\sigma(1)}}\left(\frac{\partial}{\partial x_1}\right)}_{\delta_{i_{\sigma(1)},1}} \cdots \underbrace{dx_{i_{\sigma(n)}}\left(\frac{\partial}{\partial x_n}\right)}_{\delta_{i_{\sigma(n)},n}}$$

$$= f$$

Then as $\omega > 0 \Rightarrow f > 0$. (this holds on U where $x = (x^i)$ are defined)

(1) Suppose $(V, y) \in \mathcal{A}_M^+$ overlapping $(U, x) \in \mathcal{A}_M^+$ that is $U \cap V \neq \emptyset$. Since $(V, y) \in \mathcal{A}_M^+$ we can find $g : V \longrightarrow (0, \infty)$ where (on V at least)

$$\omega = g \, dy_1 \wedge dy_2 \wedge \cdots \wedge dy_n$$

If $p \in U \cap V$ then $d_p y_j = \frac{\partial y_j}{\partial x_k}(p) \, d_p x_k$ thus,

$$\begin{aligned}
\omega_p &= g(p) \, d_p y_1 \wedge d_p y_2 \wedge \cdots \wedge d_p y_n \\
&= g(p) \left(\frac{\partial y_1}{\partial x_{i_1}}\Big|_p d_p x_{i_1} \right) \wedge \left(\frac{\partial y_2}{\partial x_{i_2}}\Big|_p d_p x_{i_2} \right) \wedge \cdots \wedge \left(\frac{\partial y_n}{\partial x_{i_n}}\Big|_p d_p x_{i_n} \right) \\
&= g(p) \frac{\partial y_1}{\partial x_{i_1}}(p) \frac{\partial y_2}{\partial x_{i_2}}(p) \cdots \frac{\partial y_n}{\partial x_{i_n}}(p) \, d_p x_{i_1} \wedge d_p x_{i_2} \wedge \cdots \wedge d_p x_{i_n} \\
&= g(p) \underbrace{\frac{\partial y_1}{\partial x_{i_1}}(p) \frac{\partial y_2}{\partial x_{i_2}}(p) \cdots \frac{\partial y_n}{\partial x_{i_n}}(p) \, \epsilon_{i_1 i_2 \cdots i_n}}_{\det\left(\left(\frac{\partial y_i}{\partial x_j}(p)\right)\right)} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \\
&= f(p) \, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n
\end{aligned}$$

Thus we can read off,

$$g(p) \det\left(\left(\frac{\partial y_i}{\partial x_j}(p)\right)\right) = f(p) \qquad \text{now } f(p), g(p) > 0$$

$$\Rightarrow \boxed{\det\left(\left(\frac{\partial y_i}{\partial x_j}(p)\right)\right) > 0 \quad \text{for } (U, x), (V, y) \in \mathcal{A}_M^+}$$

It remains to show $\mathcal{A}_M^+$ is an atlas.

(1) Need to show $\mathcal{A}_M^+$ is a subatlas of $\mathcal{A}_M$.

Note that since $\omega \neq 0$ on M it follows that for each $(U, x) \in \mathcal{A}_M$ we have

$$\text{I.)}\ \omega\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) > 0$$

$$\text{II.)}\ \omega\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) < 0$$

Suppose (U, x) falls into case II. We note $\exists\ (U, \tilde{x}) \in \mathcal{A}_M$ with $(U, \tilde{x}) \in \mathcal{A}_M^+$ as follows, define $\tilde{x}$ by

$$\tilde{x}(p) \equiv \left(x_2(p), x_1(p), x_3(p), \dots, x_n(p)\right)$$

then

$$\begin{aligned} d\tilde{x}_1 \wedge d\tilde{x}_2 \wedge d\tilde{x}_3 \wedge \cdots \wedge d\tilde{x}_n &= dx_2 \wedge dx_1 \wedge dx_3 \wedge \cdots \wedge dx_n \\ &= -dx_1 \wedge dx_2 \wedge dx_3 \wedge \cdots \wedge dx_n \end{aligned}$$

thus,

$$\begin{aligned} \omega &= f\, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \quad (\text{with } f(U) < 0) \\ &= -f\, d\tilde{x}_1 \wedge d\tilde{x}_2 \wedge \cdots \wedge d\tilde{x}_n \end{aligned}$$

Hence in $(\tilde{x})$ will have,

$$\omega_p\left(\frac{\partial}{\partial \tilde{x}_1}\Big|_p, \frac{\partial}{\partial \tilde{x}_2}\Big|_p, \dots, \frac{\partial}{\partial \tilde{x}_n}\Big|_p\right) = -f(p) > 0 \quad \Rightarrow \text{type I.}$$

The Jacobean of $\tilde{x} \circ x^{-1}$ will be of the form

$$J_{\tilde{x}\circ x^{-1}} = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \Rightarrow \det(J_{\tilde{x}\circ x^{-1}}) = -1 \neq 0 \ \therefore (U, \tilde{x}) \in \mathcal{A}_M.$$

((U, $\tilde{x}$) is compatible with (U, x) ∴ it is included in Maximal atlas $\mathcal{A}_M$)

Thus it is clear $\mathcal{A}_M^+$ forms subatlas as there are type I. charts everywhere.

(2) The integral of $f: M \longrightarrow \mathbb{R}$ is defined,

Def^{n} (5.7) Let M be a manifold with volume form ω and $f: M \longrightarrow \mathbb{R}$ a function which is continuous with compact support, Then

$$\int_M f \equiv \int_M f \cdot \omega$$

Remark (5.8) If M is oriented by ω and $-M$ is M oriented by $-\omega$ then for $\alpha \in \Omega^n_c(M)$ (α with compact support on M)

$$-\int_M \alpha = \int_{-M} \alpha$$

Thus if $\varphi: N \longrightarrow M$ is a diffeomorphism,

$$\int_M \alpha = \int_N \varepsilon \cdot \varphi^* \alpha$$

where ε is locally constant with value ± 1 according to whether φ locally preserves or reverses orientation

Claim: the integral of functions over an oriented manifold is independent of choice of orientation. Let f have compact support,

$$\int_{-M} f \underset{\substack{\text{def}^n \text{ of} \\ \text{fnct. integration} \\ \text{on } (M, -\omega)}}{\equiv} \int_{-M} f \cdot (-\omega) = -\int_{-M} f \cdot \omega \underset{\substack{\text{remark} \\ 5.8}}{=} -\left(-\int_M f \cdot \omega\right) \underset{\substack{\text{def}^n \\ \text{of fnct.} \\ \text{integration} \\ \text{on } (M, \omega)}}{\equiv} \int_M f$$

Next we prove Remark 5.8,

(2) continued, proof of Remark 5.8, following [BD]

Since Supp(α) is compact $\Rightarrow$ can cover with finitely many chart domains. Then using the partition of unity

$$\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_k$$

where Supp$(\alpha_i) \subset U_i \leftarrow$ chart domain. Then the integral of a form which is nonzero on more than just one chart domain has a natural definition, we sum the contributions of each α_i together

$$\int_M \alpha = \sum_k \int_{U_k} \alpha = \sum_k \int_{U_k} \alpha_k \qquad \left(\begin{array}{l} \alpha_i = 0 \quad i \neq k \\ \text{on } U_k \text{ by} \\ \text{def}^{\underline{n}} \text{ of} \\ \text{support} \end{array}\right)$$

prop. 5.5 explains all of this in more detail.

Note $\varphi : N \to M$ induces charts on N from those on M by the pull-back ; $\varphi^* x_i = x_i \circ \varphi : N \to M \to \mathbb{R}^n$.
On each chart domain U_i the diffeomorphism φ either preserves or reverses the orientation (if it did both then somewhere $\det(D\varphi) = 0$ hence φ fails to be diffeomorphism).
Because M may be disconnected it is possible for φ to be somewhere orientation preserving and elsewhere reversing.
Thus we see (assuming $-\int_{U_k} \alpha_k = \int_{-U_k} \alpha_k \quad \alpha_k \in \Omega^n_c(U_k)$)

$$\int_M \alpha = \int_N \varepsilon \cdot \varphi^* \alpha \qquad (\mathrm{dom}(x_k \circ \varphi).)$$

where $\varepsilon = \pm 1$ over the chart domains $\varphi^{-1}(U_k)$ on N.

It suffices to check $-\int_M \alpha = \int_{-M} \alpha$ for $\mathrm{supp}(\alpha) \subset U = \mathrm{dom}(x)$.

(2) Let $\operatorname{supp}(\alpha) \subset U = \operatorname{dom}(h)$ for $(U, h) \in \mathcal{a}_m$ oriented by ω. Now $h : U \longrightarrow U' \subseteq \mathbb{R}^n$

$$\int_M \alpha = \int_U \alpha$$: since $\operatorname{supp}(\alpha) \subset U$ its zero on M outside U.

$$= \int_{U'} (h^{-1})^* \alpha$$: def$^{\underline{n}}$ of form-integration.

$$= \int_{U'} a \, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$$: using (x_i) for coordinates & parameters on $\mathbb{R}^n$

$$= \int_{U'} a(x) \, dx_1 \, dx_2 \cdots dx_n$$

Now reverse orientation by transformation $\psi : U' \longrightarrow \mathbb{R}^n$

$$\psi(x_1, x_2, \ldots, x_n) = (-x_1, x_2, \ldots, x_n)$$

Then, its clear $\psi \circ h$ gives orientation $-\omega$ on M if h gave ω,

$$\int_{-M} \alpha = \int_{\psi U'} (\psi h)^{-1*} \alpha$$

$$= \int_{\psi U'} (\psi^{-1})^* (h^{-1})^* \alpha$$: prop. of pull-backs.

$$= \int_{\psi U'} (\psi^{-1})^* [a \cdot dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n]$$

$$= \int_{\psi U'} (\psi^{-1})^*(a) \, (\psi^{-1})^* (dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n)$$

a is zero-form and $\ell^*(\alpha \wedge \beta) = (\ell^* \alpha) \wedge (\ell^* \beta)$

$$= \int_{\psi U'} (a \circ \psi^{-1}) \cdot (-dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n)$$

$$= -\int_{\psi U'} a(\psi^{-1}(x_1, x_2, \ldots, x_n)) \, dx_1 \, dx_2 \cdots dx_n$$

$\psi \circ \psi = id$. $\therefore \underline{\psi = \psi^{-1}}$

$$= -\int_{\psi U'} a(-x_1, x_2, \ldots, x_n) \, dx_1 \, dx_2 \cdots dx_n$$

$$= -\int_{U'} a(u_1, u_2, \ldots, u_n) \Big| \underbrace{\det(D\psi^{-1})}_{-1} \Big| du_1 \cdots du_n = -\int_{U'} a(u) \, du_1 \cdots du_n$$

(2) Concluding,

$$\int_{\overline{M}} \alpha = -\int_{U'} a(u)\, du_1\, du_2 \cdots du_n$$

$$= -\int_{U'} a(x)\, dx_1\, dx_2 \cdots dx_n \quad \text{: relabling dummies.}$$

$$\left.\begin{aligned} &= -\int_{U'} (h^{-1})^* \alpha \\ &= -\int_{U} \alpha \\ &= -\int_{M} \alpha \end{aligned}\right\} \text{ just reversing earlier steps.}$$

Finally we pause to check that if h orients ω then ψh orients $-\omega$.

$$h \equiv (y_1, y_2, \ldots, y_n)$$
$$\psi h = (-y_1, y_2, \ldots, y_n)$$

$$\omega\left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \ldots, \frac{\partial}{\partial y_n}\right) > 0 \quad \text{by assumption}$$

$$\omega\left(\frac{-\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \ldots, \frac{\partial}{\partial y_n}\right) = -\omega\left(\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\right) < 0$$

$\Rightarrow \psi \circ h$ has orientation $-\omega$.

(3) Th^{m} (5.13)

Let G be a compact Lie group and $C(G)$ the real vector space of continuous fncts. on G. The invariant integral $f \longmapsto \int f(g)\,dg$ is uniquely determined by

(i.) It is linear, monotone and normalized ($\int 1 = 1$)
(ii) Left invariance, $\int f \circ l_h = \int f$ for any $h \in G$

Proof: Suppose $f \longmapsto \int f(h)\,\delta h$ denotes an integral with those properties (i), (ii). Then let $f \longmapsto \int f(g)\,dg$ be a particular right-invariant integral, so $\int f(gh)\,dg = \int f(g)\,dg$. We construct this right-inv. integral,

$$\omega_i = l_g^* \omega_i \qquad \& \qquad \bar{\omega}_i = r_g^* \omega_i \qquad i = 1, 2, \ldots, n$$

$$\Omega = \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n \qquad \bar{\Omega} = \bar{\omega}_1 \wedge \bar{\omega}_2 \wedge \cdots \wedge \bar{\omega}_n$$

$$l_g^* \Omega = \Omega \qquad \underbrace{r_g^* \bar{\Omega} = \Omega}$$

this will construct a particular right-invariant integral, the particularity ultimately linked to the basis of $T_e^* G$ which is $\{(\bar{\omega}_1)_e, (\bar{\omega}_2)_e, \ldots, (\bar{\omega}_n)_e\}$ these are then pulled-back all over the group to give the $\bar{\omega}_i$'s.

$$\int f(gh)\,dg = \int f \circ r_h \equiv \int (f \circ r_h) \cdot \bar{\Omega} = \int r_h^* f \, r_h^* \bar{\Omega}$$

$$= \int r_h^* (f \bar{\Omega}) = \int f \bar{\Omega} \equiv \int f(gh)\,dg$$

So we've constructed a particular right-invariant integral, indeed the "group is sufficiently symmetric" to gaurantee its existence.

(3) Th^{m} (5.13) continued

Consider $f : G \times G \longrightarrow \mathbb{R}$ then define,

$$\int f(g,h) \equiv \int\left(\int f(g,h)\,dg\right)\delta h$$

Claim: $\int f(g,h) = \int\left(\int f(g,h)\,\delta h\right)dg.$

Suppose $f(g,h) = \varphi(h)\psi(g)$ then,

$$\begin{aligned}
\int f(g,h) &= \int\left(\int \varphi(h)\psi(g)\,dg\right)\delta h \\
&= \int \varphi(h)\left(\int \psi(g)\,dg\right)\delta h \\
&= \int\left(\int \psi(g)\,dg\right)\varphi(h)\,\delta h \\
&= \int \psi(g)\,dg \int \varphi(h)\,\delta h \\
&= \int\left(\int \varphi(h)\,\delta h\right)\psi(g)\,dg \\
&= \int\left(\int \varphi(h)\psi(g)\,\delta h\right)dg
\end{aligned}$$

- To accomplish the calculation above we've used ~~that $\int dg$ & $\int \delta h$~~ give real number outputs, and we can "clearly" pull out functions which are independent of integration variable
- Thus the Fubini-type Th^{m} holds for fncts. of g & h which are of the form $f(g,h) = \varphi(h)\psi(g)$.

Why are the functions

$$f(g,h) = \varphi(g)\psi(h)$$

for some $\varphi, \psi \in C(G)$ dense in $C(G \times G)$ under the uniform convergence topology? This result apparently stems from Stone-Weirstrauss Th^{m}. Lets see how.

$\underline{\text{Th}^m (2.7)}$ Let $\mathbb{X}$ be compact, and use sup-norm on C^0, (i.) Let $B \subset C^0(\mathbb{X}, \mathbb{R})$ be a subalgebra which contains all real constants and seperates points. Then B is dense in $C^0(\mathbb{X}, \mathbb{R})$

"seperates points" means given $x_1, x_2 \in \mathbb{X}$ which are distinct $(x_1 \neq x_2)$ then $\exists f \in B$ such that $f(x_1) \neq f(x_2)$.

Let us consider a compact Lie group G. Then $G \times G$ is also compact. Then define

$$B \equiv \{ f \in C(G \times G) \mid f(g,h) = \varphi(g)\psi(h) \text{ for } \varphi, \psi \in C(G) \}$$

we seek to show B is a subalgebra seperating points.

$C(G)$ is an algebra under point-wise multiplication and ~~addition~~ of functions. The functions are real-valued so the following observations are clear. Suppose $f_1, f_2 \in B$

$$\begin{aligned}(f_1 f_2)(g,h) &= \varphi_1(g)\psi_1(h)\varphi_2(g)\psi_2(h) \\ &= \varphi_1(g)\varphi_2(g)\psi_1(h)\psi_2(h) \\ &= (\varphi_1\varphi_2)(g)(\psi_1\psi_2)(h) \qquad \therefore f_1 f_2 = (\varphi_1\varphi_2)\cdot(\psi_1\psi_2)\end{aligned}$$

$$\therefore f_1 f_2 \in B$$

$\underline{\text{remark:}}$ it seems B doesn't close under addition,

$$(\varphi_1\psi_1 + \varphi_2\psi_2)(g,h) = \varphi_1(g)\psi_1(h) + \varphi_2(g)\psi_2(h) \Rightarrow f_1 + f_2 \in \text{Span}(B)$$

We should make B the span of B defined above otherwise B not a linear space.

(3) (5.13) Need to show $B \subset C(G\times G, \mathbb{R})$ is dense.
we saw B is subalgebra of $C(G\times G)$ now
we need to show B seperates points in $G\times G$

Let $(x_1, y_1), (x_2, y_2) \in G\times G$ such that

$$(x_1, y_1) \neq (x_2, y_2)$$

we seek to find $f \in B$ such that $f(x_1, y_1) \neq f(x_2, y_2)$

To begin we divide into three cases,

I.) $x_1 = x_2$ but $y_1 \neq y_2$

II.) $x_1 \neq x_2$ but $y_1 = y_2$

III.) $x_1 \neq x_2$ and $y_1 \neq y_2$

In each case we will construct an appropriate $f(x,y)$.

I.) Let $f(x,y) = e(x)\, l_a(y)$ where $e(x) = e$, $l_a(y) = ay$, $a \in G$

$$f(x_1, y_1) = e(x_1)\, l_a(y_1) = e a y_1 = a y_1$$
$$f(x_2, y_2) = e(x_2)\, l_a(y_2) = e a y_2 = a y_2$$

then $y_1 \neq y_2 \Rightarrow a y_1 \neq a y_2 \therefore f(x_1, y_1) \neq f(x_2, y_2)$ *

II.) Likewise $f(x,y) = l_a(x)\, e(y)$ then

$$x_1 \neq x_2 \Rightarrow a x_1 \neq a x_2 \Rightarrow f(x_1, y_1) \neq f(x_2, y_2)$$

III.) Either previous construction will work.

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* Sadly I realize this argument is not the need one. Our functions go into $\mathbb{R}$ not G so no constant map to $e = id_G$ or left multiplication by a is available. I can't see how to construct $f(x,y)$ correctly at this time, we'll assume its possible and go on,

(3) <u>thoughts on 5.13 continued</u>

$$\mathcal{B} \equiv \{ f(g,h) = \varphi(g)\psi(h) \mid \varphi, \psi \in C(G) \} \subseteq C(G\times G)$$

is <u>dense</u> in $C(G\times G)$ by Stone-Weierstrass.

This means that in each nbhd of $C(G\times G)$ there is at least one point from $\mathcal{B}$ contained.

Equivalently for any $f \in C(G\times G)$ $\exists$ $\{\varphi_n \psi_n\}_{n=1}^{\infty} \subset \mathcal{B}$ such that

$$\varphi_n \psi_n \longrightarrow f \quad \text{as } n \to \infty$$

$$\Rightarrow \iint \varphi_n \psi_n \longrightarrow \iint f$$

$$\Rightarrow \int f(g,h) = \iint f \, dg \, dh \underset{(*)}{=} \iint f \, dh \, dg$$

a little detail missing here.

since $(*)$ holds for each n.

(3) Proof of 5.13 continued

Let $f : G \to \mathbb{R}$ be continuous consider $\tilde{f}(g,h) = f(gh)$, clearly this is continuous on $G \times G$

$$\int f(g)\,dg = \int f(g)\,dg \int \delta h$$: (we scale $\overline{\Omega}$ so that $\int_G \overline{\Omega} = 1$. $\Rightarrow \int \delta h = 1$.)

$$= \int\left(\int f(g)\,dg\right)\delta h$$: brought in real # into integral over δh

$$= \int\left(\int f(gh)\,dg\right)\delta h$$: by right invariance.

$$= \int\left(\int \tilde{f}(g,h)\,dg\right)\delta h$$

$$= \int\left(\int \tilde{f}(g,h)\,\delta h\right)dg$$: using Fubini result just proved

$$= \int\left(\int f(gh)\,\delta h\right)dg$$

$$= \int\left(\int f(h)\,\delta h\right)dg$$: using left-invariance.

$$= \int f(h)\,\delta h \int dg$$

$$= \int f(h)\,\delta h$$ $\therefore$ the arbitrary left-inv. int. $\int f(h)\delta h$ is equal to the particular right invariant $\int f(g)\,dg$. This proves uniqueness.

remark: the existence of a left-invariant integral will be shown in problem (4) in almost the same way as we constructed $\int_G f(g)\,dg = \int_G f \cdot \overline{\Omega}$ here.

4) Wish to prove

$$\int f(g)\,dg \underset{(i)}{=} \int f(hg)\,dg \underset{(ii)}{=} \int f(gh)\,dg \underset{(iii)}{=} \int f(g^{-1})\,dg \underset{(iv)}{=} \int (f\circ\varphi)(g)\,dg$$

Proof of (i)

$$\begin{aligned}
\int_G f(hg)\,dg &= \int (f\circ l_h)(g)\,dg \\
&\equiv \int (f\circ l_h)\,\Omega \\
&= \int (f\circ l_h)(l_h^*\Omega) \\
&= \int (l_h^*\circ f)(l_h^*\Omega) \\
&= \int l_h^*(f\Omega) \\
&= \int f\Omega \\
&= \int f(g)\,dg
\end{aligned}$$

using left invariance of Ω.

using $\int_M \alpha = \int_N \varepsilon\cdot\varphi^*\alpha$, $(\varepsilon = \pm 1)$

$\varphi : N \longrightarrow M$ a diffeomorphism

in our case l_h is such a map.

l_h is orientation preserving over all of G

The proof of (ii), (iii), (iv) follow from an application of 5.13 note $f \longmapsto \int f\circ\varphi$ is linear and monotone and normalized. If we can show left-invariance $\int f\circ l_h\circ\varphi = \int f\circ\varphi$ we may conclude this is the unique left-invariant integral that is $\int f = \int f\circ\varphi$. Particular choices of φ will yield the desired results.

Proof of (ii)

Let $\varphi(g) = R_h(g) = gh$. Consider

$$\int (f \circ l_k \circ \varphi)(g)\,dg = \int f(kgh)\,dg$$
$$= \int f(gh)\,dg$$
$$= \int (f \circ R_h)(g)\,dg$$
$$= \int f \circ \varphi \quad \therefore \int f \circ R_h = \int f$$

$$\therefore \int f(gh)\,dg = \int f(g)\,dg$$

(and $\int f(g)\,dg = \int f(hg)\,dg$ is known)

Proof of (7(i'))

Consider $\varphi(g) = g^{-1}$,

$\int f(g^{-1}h)\,dg = \int f(hg^{-1})\,dg$

$\int f\circ l_h \circ \varphi = \int f(hg^{-1})\,dg = \int f(g^{-1}h)\,dg$, (since $\int (f\circ Inv)(h^{-1}g)\,dg = \int (f\circ Inv)(g^{-1}h)\,dg$

$= \int f\left((h^{-1}g)^{-1}\right)dg \qquad (h^{-1}g)^{-1} = g^{-1}h$

$= \int (f\circ Inv)(h^{-1}g)\,dg$

$= \int (f\circ Inv \circ l_{h^{-1}})(g)\,dg$

$= \int (f\circ Inv)(g)\,dg$

using left invariance of $\int f\circ Inv$.

$= \int f\circ\varphi \qquad \therefore \int f\circ\varphi = \int f \qquad \therefore \int f(g^{-1})\,dg = \int f(g)\,dg$

Proof of (iv)

Let $\varphi : G \to G$ be an automorphism.
Let φ play the same role as φ in 5.13,

$$\int (f \circ l_h \circ \varphi)(g)\,dg = \int (f \circ l_h \circ \varphi \circ l_a)(g)\,dg \quad \text{by left-inv.}$$

Notice that

$$(l_h \circ \varphi \circ l_a)(x) = \varphi(x)$$

$$h\varphi(ax) = \varphi(x)$$

$$\varphi(ax) = h^{-1}\varphi(x)$$

$$ax = \varphi^{-1}(h^{-1}\varphi(x))$$

$$a = \varphi^{-1}(h^{-1}\varphi(x))x^{-1} = \varphi^{-1}(h^{-1})\varphi^{-1}(\varphi(x))x^{-1} = \varphi^{-1}(h^{-1})xx^{-1}$$

$$\therefore \underline{l_h \circ \varphi \circ l_{\varphi^{-1}(h^{-1})} = \varphi}$$

Thus $\int f \circ l_h \circ \varphi = \int f \circ l_h \circ \varphi \circ l_{\varphi^{-1}(h^{-1})}$

$$= \int f \circ \varphi$$

$$\therefore \underline{\int f(g)\,dg = \int (f \circ \varphi)(g)\,dg}$$