$$\begin{cases} M & \text{manifold} \\ \mathcal{D} & \text{distribution which is involutive} \\ d = \dim(\mathcal{D}(x)), & m = \dim M \\ p \in M & \end{cases}$$

$\{\exists\ (V, x)$ chart at p such that plaques of (V, x) are integral manifolds of $\mathcal{D}$

Proof: We proved this for $d = 1$. Assume it is true for M of dimension $m-1$ and $\mathcal{D}$ of dim $d-1$. (Assume true for all manifolds with the assumptions)

Let M be dim m and $\mathcal{D}$ dim d and $p \in M$. Since $\mathcal{D}$ smooth $\exists\ \tilde{V}$ open about p and vector fields

$$X_1, X_2, \ldots, X_d \quad \text{on} \quad \tilde{V}$$

such that

$$\mathcal{D}(q) = \langle X_1(q), X_2(q), \ldots, X_d(q) \rangle$$

$\forall q \in \tilde{V}$ by prev. Th^{m} (on A6) $\exists$ a chart (V, y) such that $p \in V \subseteq \tilde{V}$ and then we can straighten out X_1

$$X_1\big|_V = \frac{\partial}{\partial y^1}\Big|_V$$

we knew $X_1 \neq 0$ since it was part of basis for $\mathcal{D}$ on V: We may assume y is cubic centered at p.

On V define vector fields

$$Y_1, Y_2, \ldots, Y_d$$

So that $Y_1 = X_1$ and also $\cancel{Y_i = X_i(y^1) X_1}$

$$\underbrace{Y_i = X_i - \underline{X_i(y^1)\, X_1}}_{} \qquad i = 2, 3, 4, \ldots$$

in total only in directions tangent to surface $y^1 =$

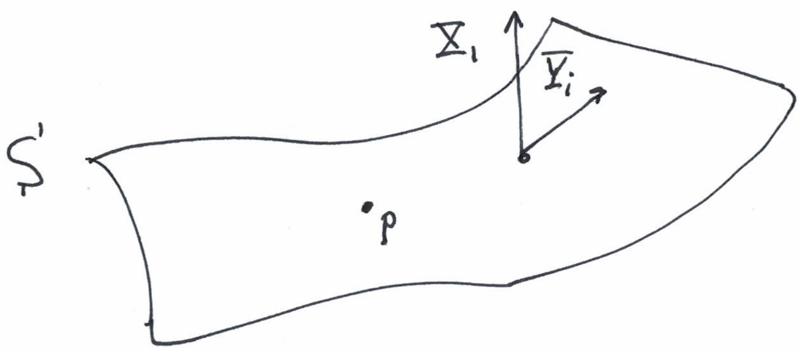
Component of X_i in the $(y^1)^{\perp}$-direction

The vector fields $X_1, X_2, \ldots, X_d$ are LI, thus $\cancel{X_1(q)}$, $Y_1(q), Y_2(q), \ldots, Y_d(q)$ is a basis of $\mathcal{D}(q)$ $\forall q \in V$.

Let us define a $(m-1)$ - subspace of $\mathcal{D}$

$$S' = \{ q \in V \mid y'(q) = 0 \}$$

Recall a vector v is tangent to $y' = 0$ iff $dy'(v) = 0$



$(\nabla y') \cdot V = 0$

n-dim'l gradient here, this is in $\mathbb{R}^n$.

Note $T_q S' = \{ v \in T_q V \mid \overbrace{dy'(v) = 0}\}$

and for $i = 2, 3, \ldots, d$ we find

$$\begin{aligned} d_q y'(Y_i(q)) &= Y_i(y')(q) \\ &= X_i(y')(q) - X_i(y')_q \overset{1}{\overset{\|}{X_1(y')_q}} \\ &= 0. \end{aligned}$$

So if we define Z_i on S' by

$$Z_i(q) = Y_i(q)$$

$\forall q \in S'$ then Z_i are vector fields on S' for $i = 2, 3, \ldots, d$. which are LI so $Z_2, Z_3, \ldots, Z_d$ span a $(d-1)$-dim'l distribution on S'

Let $\mathcal{D}_S(q)$ denote the subspace $T_q S'$ spanned by $Z_2(q), Z_3(q), \ldots, Z_d(q)$. (In fact $\mathcal{D}_S(p) = T_p S$)

We claim that $\mathcal{D}_S$ is involutive. Let

$$i : S \hookrightarrow V$$

be the inclusion map. For $j = 2, 3, \ldots, d$ we have Z_j is i-related to Y_j therefore

$$[Z_j, Z_k] \overset{i}{\sim} [Y_j, Y_k]$$

for $j, k = 2, 3, \ldots, d$. But since $Y_j(y') = dy'(Y_j) = 0$ observe,

$$\begin{aligned} dy'([Y_j, Y_k]) &= [Y_j, Y_k](y') \\ &= Y_j(Y_k(y')) - Y_k(Y_j(y')) \\ &= Y_j(0) - Y_k(0) \\ &= 0 \end{aligned}$$

We'd like to transfer this over to the Z's. Note

$$[Y_j, Y_k] = \sum_{i=1}^{d} C_{jkl} Y_l$$

Apply dy' to both sides,

$$0 = dy'[Y_j, Y_k] = \sum_{i=1}^{d} C_{jkl} \underbrace{dy'(Y_l)}_{\text{zero for } l=2,\ldots,d} = C_{jk1}\, dy'(Y_1) = C_{jk1}$$

Therefore we can start the Σ from $i=2$,

$$(\star) \quad [Y_j, Y_k] = \sum_{l=2}^{d}{}' C_{jkl} Y_l \qquad j, k = 2, 3, \ldots, d.$$

restriction to S' of (✯) 10/27/06.04

By the ~~i-relatedness~~ we may conclude that

$$[Z_j, Z_k] = \sum_{l=2}^{d} (c_{jkl}|_S) Z_l$$

Therefore $\mathcal{D}_S$ is involutive. If we take

$$\left.\begin{aligned} W_1 &= \sum_{j=2}' \lambda_j Z_j \\ W_2 &= \sum_{k=2}' \mu_k Z_k \end{aligned}\right\} \quad [\lambda_j Z_j, \mu_k Z_k]$$

eventually get terms like $\lambda_j \mu_k [Z_j, Z_k]$ and other terms with just Z_k or Z_j but since $[Z, Z] \in \mathcal{D}_S$

Thus $\mathcal{D}_S(q) = \langle Z_2(q), Z_3(q), .., Z_d(q) \rangle \ \forall q \in S'$.

By induction $\exists$ a chart $\tilde{w}$ defined on a nbhd $\tilde{\Theta}$ of p in $S' \ni \tilde{w}$ is cubic is centered at p and plaques of $(\tilde{\Theta}, \tilde{w})$ are integral manifolds of $\mathcal{D}_S$.

Begin again next time