REPRESENTATION THEORY OF LIE GROUPS

p65 Def^n/ A reprentation of a Lie group G is a continuous mapping

(1.) $\ell : G \longrightarrow Aut(V)$ a homomorphism.

(2.) $\rho : G \times V \longrightarrow V$ with $\rho(g_1, \rho(g_2, v)) = \rho(g_1 g_2, v)$
$\rho(e, v) = v$

(3.) $g \cdot v = \rho(g, v)$

here (2.) & (3.) are same and (1.) $g \cdot v = \rho(g, v) = \ell(g)(v)$

Remark: Aut V is given standard topolgy (not Zariski).

- A matrix representation is a homomorphism $\ell : G \longrightarrow Gl(n)$, this is where the theory begun later $Aut(V)$ or $Aut(\mathbb{C}^n) \cong Gl(n, \mathbb{C})$.

Def^n/ A representation $\ell : G \longrightarrow Aut(V)$ is faithful iff $\ker \ell = \{e\}$.

Every compact Lie group has a faithful representation. (Peter-Wyle $\approx$)

p 66-67 he defines convolutions, not on shortest path we go on.

p67 Def^n/ A morphism between two representations V, W is a linear map $f : V \longrightarrow W$ such that

$$f \circ \ell_g^V = \ell_g^W \circ f$$

where ℓ_g^V, ℓ_g^W are rep. of G associated to V & W respective.

- here one often says such f is equivariant or intertwining operator in other books. Also other notations,

$$f(\rho_V(g, v)) = \rho_W(g, f(v))$$
$$f(g \cdot v) = g \cdot f(v)$$

Def^n/ $Hom_G(V, W)$ = set of morphisms from V to W.

- We say $\rho_V \cong \rho_W$ or $V \cong W$ iff $\exists$ a morphisms f, f^{-1} that are vector space isomorphisms of V & W.

- this all works for topological groups although Haar measure is hard to prove existence of.

When are matrix representations equivalent?

$$\alpha : G \longrightarrow Gl(n)$$
$$\beta : G \longrightarrow Gl(n)$$

$$\alpha \cong \beta$$

$$f(\beta(g)(x)) = \alpha(g)(f(x))$$

$$A_f \cdot \beta(g) \cdot x = \alpha(g) \cdot A_f \cdot x$$

$$\boxed{\beta(g) = A_f^{-1} \alpha(g) A_f} \quad \forall g$$

V complex vector space, assume V has a Hermitian inner product:

$$(x, y) \longmapsto \langle x, y \rangle$$

$$\left.\begin{aligned} \langle ax + y, z \rangle &= a \langle x, z \rangle + \langle y, z \rangle \\ \langle x, ay + z \rangle &= \bar{a} \langle x, y \rangle + \langle x, z \rangle \end{aligned}\right\}$$

conjugate linear map from $V \times V \to \mathbb{C}$.

Other properties,

$$\overline{\langle x, y \rangle} = \langle y, x \rangle$$
$$\langle x, x \rangle \geq 0$$
$$\langle x, x \rangle = 0 \Rightarrow x = 0$$

If G acts a Hermitian inner product space $(V, \langle \, , \rangle)$ then the inner product is G-invariant if

$$\langle g \cdot x, g \cdot y \rangle = \langle x, y \rangle$$

for all $g \in G$, $x, y \in V$

Remark: most if not all of this works on Hilbert space, the finite dimensionality of V is non-essential.

Thm/ If $\alpha : G \to \text{Aut}\, V$ is a rep. of a compact group and V is a Hermitian inner product space then V admits a <u>G-invariant inner product</u>

<u>Remark</u>: our takehome final is to develop this invariant integral to do it. Also consider M is orientable then $\exists$ everywhere nontrivial volume form. On a Lie group G one can consider left invariant one-forms on G. a one-form on G is left invariant iff

$$l_g^* \alpha = \alpha$$

there existence follows from left-translation of $T_e^* G$, simply define $\alpha_g \equiv l_g^* \alpha_e$. Take basis $\alpha_1, \alpha_2, \dots, \alpha_n$ of $T_e^* G$ then

$$l_g^* (\alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_n) = (l_g^* \alpha_1) \wedge (l_g^* \alpha_2) \wedge \dots \wedge (l_g^* \alpha_n)$$
$$= \alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_n$$

this is a left-invariant volume on the Lie group. Marsden or Loomis show how to extend integral of forms with compact support to more abstract cases. Also Varadarajan is good.

<u>Proof</u>: $\alpha(g)(v) = g \cdot v$

$$b(u,v) \equiv \int_G \langle g\cdot u, g \cdot v \rangle dg$$

$$\overline{b(u,v)} = \int_G \overline{\langle g\cdot u, g\cdot v\rangle}\, dg = \int_G \langle g\cdot v, g\cdot u\rangle dg = b(v,u)$$

b is conjugate linear since $\langle , \rangle$ is conjugate linear and the group action is linear as well.

$$b(u,u) = \int_G \langle g\cdot u, g\cdot u\rangle dg$$

$\langle g\cdot u, g\cdot u\rangle > 0$ ($u \neq 0$) $\Rightarrow$ integral of positive is likewise positive as integral is over compact group.

Proof continued:

$$b(hu, hv) = \int_G \langle ghu, ghv \rangle \, dg$$

$$= \int_G \langle ghu, ghv \rangle \, d(gh)$$

left invar. $\Rightarrow$ right inv. on compact Lie group.

$$= \int_G \langle \bar{g} u, \bar{g} v \rangle \, d\bar{g}$$

$$= b(u, v)$$

Theorem V, W representation spaces.

(i) χ_V is smooth

(ii) $V \cong W \implies \chi_V = \chi_W$

(iii) $\chi_V(g^{-1}xg) = \chi_V(x)$

(iv) $\chi_{V\oplus W} = \chi_V + \chi_W$

(v) $\chi_{V^*}(g) = \chi_V(g^{-1})$

(vi) $\chi_{V\otimes W} = \chi_V \chi_W$

(vii) $\chi_{\overline{V}}(g) = \overline{\chi_V(g)} = \chi_V(g^{-1})$

(viii) $\chi_V(e) = \dim_{\mathbb{C}} V$

We know (p. 29)

If $\varphi : G \to H$ is a continuous homomorphism then φ is smooth.

Proof (i) If V is a rep. of G then

$$g \longrightarrow {}^V M^i_j(g) \text{ is smooth.}$$

$$g \longrightarrow \sum_i {}^V M^i_i(g) \text{ is smooth}$$

ii) $f: V \to W$ isomorphism

$$(l_g^W \circ f)(x) = g \cdot f(x)$$
$$\cancel{= (gf)(x)}$$
$$= f(g \cdot x)$$
$$= (f \circ {}^V l_g)(x)$$

$$\chi_V(g) = Tr({}^V l_g) = Tr(f^{-1} \circ l_g^W \circ f)$$
$$= Tr(l_g^W) = \chi_W(g)$$

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v) $$\chi_{V \oplus W}(g) = Tr(l_g^{V \oplus W})$$
$$= Tr\begin{pmatrix} l_g^V & 0 \\ 0 & l_g^W \end{pmatrix}$$
$$= Tr(l_g^V) + Tr(l_g^W)$$
$$= \chi_V(g) + \chi_W(g)$$

(vi) G acts on $V \otimes W$

$g\cdot(v \otimes w) = (g \cdot v) \otimes (g \cdot w)$

$\rho_g^{V\otimes W}(v \otimes w) = \rho_g^V(v) \otimes \rho_g^W(w)$

$\{v_i\}$ basis of V $\{w_i\}$ basis of W

then $\rho_g(v_i \otimes w_j) = \rho_g^V(v_i) \otimes \rho_g(w_j)$

recall $g \cdot v_j = M_j^i(g)\, v_i$

so

$= {}^V M_i^k(g)\, v_k \otimes {}^W M_j^l(g)\, w_l$

$= {}^V M_i^k(g)\, {}^W M_j^l(g) (v_k \otimes w_l)$

$\chi_{V\otimes W}(g) =$

So now $\mathrm{Tr}(\rho_g^{V\otimes W}) = \mathrm{Tr}({}^V M_i^k(g)\, {}^W M_j^l(g))$

$= \mathrm{Tr}({}^V M_i^k(g))\, \mathrm{Tr}({}^W M_j^l(g))$

$= \chi_V(g)\, \chi_W(g)$

G acts on V

H acts on W

$G \times H$ acts on $V \otimes W$

$$(g,h)\cdot(v \otimes w) = (gv) \otimes (hw)$$

and $\chi_{V \otimes W}(g,h) = \chi_V(g)\,\chi_W(h)$

(vi) $\rho_g^{V^*} : V^* \to V^*$

$$\rho_g^{V^*}(\alpha)(x) \overset{\text{def}}{=} \alpha(g^{-1}x)$$

$$= \alpha(\rho_{g^{-1}}^V \cdot x)$$

$$\Rightarrow \quad \rho_g^{V^*}(\alpha) = \alpha \circ \rho_{g^{-1}}^V = (\rho_{g^{-1}})^*(\alpha)$$

recall $\quad Tr(f) = Tr(f^*)$

$$\chi_{V^*}(f) = Tr(\rho_g^{V^*}) = Tr((\rho_{g^{-1}}^V)^*) = Tr(\rho_{g^{-1}}^V)$$

$$= \chi^V(g^{-1})$$

Theorem G compact

(i) $\int_G \chi_V(g)\,dg = \dim V^G$

(ii) $\langle \chi_W, \chi_V \rangle = \int_G \chi_W(g)\,\overline{\chi_V}(g)\,dg = \dim \mathrm{Hom}_G(V,W)$

(iii) If V and W are irreducible then

$$\int_G \chi_W \overline{\chi_V} = \begin{cases} 1 & V \cong W \\ 0 & V \not\cong W \end{cases}$$

Proof

let $p: V \to V^G$ defined by ~~p(v)~~

(i) $p(v) = \int_G (gv)\,dg$ or $p = \int_G \rho_g^V\,dg$

p is projection onto fixed point set onto V^G.

Assume $V = V^G \oplus (V^G)^\perp$ relative to invariant $\langle \cdot,\cdot \rangle$

then $p = \begin{pmatrix} I_{V^G} & 0 \\ 0 & 0 \end{pmatrix}$ ~~I_{V^G} is the identity~~ I_{V^G} is the id on V^G.

Then $\mathrm{Tr}(p) = \dim V^G$

$\Rightarrow \dim V^G = \mathrm{Tr}(p) = \mathrm{Tr}\left(\int_G \rho_g^V\,dg\right)$

$= \int_G \mathrm{Tr}(\rho_g^V)\,dg = \int_G \chi_V(g)\,dg$ ✓

Pick an orthonormal basis $\{v_i\}$

$$P(v_i) = \int_G (g \cdot v_i)\, dg = \int_G M_i^j(g) v_j\, dg$$

$$= \left(\int_G M_i^j(g)\, dg\right) v_j$$

$\Rightarrow$

$$P_i^j = \int_G M_i^j(g)\, dg$$

(ii)

$$\dim \mathrm{Hom}_G(V,W) = \dim \left(\mathrm{Hom}(V,W)^G\right)$$

$$= \int_G \chi_{\mathrm{Hom}_G(V,W)}\, dg$$

$$= \int_G \chi_{V^* \otimes W}(g)\, dg$$

$$= \int \chi_{V^*}(g) \chi_W(g)\, dg = \int \chi_V(g^{-1}) \chi_W(g)\, dg$$

$$= \int \overline{\chi_V(g)}\, \chi_W(g)\, dg = \langle \chi_V(g), \chi_W(g) \rangle$$

(iii) By Schur's lemma

$$\mathrm{Hom}_G(V,W) = 0 \qquad V \not\cong W$$
$$\mathrm{Hom}_G(V,W) \cong \mathbb{C}\, \mathrm{id}_V \qquad V \cong W$$

$\square$