Representation Theory Cont'd

$\rho: G \longrightarrow Aut(V)$

$\rho: G \times V \longrightarrow V$

$g \cdot v = \rho(g, v)$

V: G-module (has a rep of G on it)

Def^n/ If $U \subseteq V$ is a subspace of V then it is a submodule iff $g \cdot u \in U$ $\forall g \in G$ and $u \in U$, we also say its a subrepresentation

$$\rho_g^U = \rho_g^V\big|_U$$

a nonzero rep. V is irreducible iff V has no submodule except 0 and V.

Prop: Let G be compact group, V a submodule of G-module U then $\exists$ a complementary submodule W such that

$$U = V \oplus W$$

moreover each G-module is direct sum of irred. submodules.

Proof: Every finite dim'l vect. space over $\mathbb{C}$ possesses inner product $\approx \mathbb{C}^n$ with $\langle z, w \rangle = \sum z_i \overline{w_i}$.

Moreover provided G-compact we can construct a G-invariant inner product (as proved last time exists) we'll denote it by $\langle , \rangle$ (as opposed to $b(,)$).

Then let $V^\perp$ denote the orthogonal complement

$$W = V^\perp = \{u \in U \mid \langle u, v \rangle = 0 \quad \forall v \in V\}$$

from linear algebra we know $U = V \oplus V^\perp$. All that remains is to check $V^\perp$ is submodule. If $u \in V^\perp$ and $g \in G$

$$\begin{aligned} \langle g \cdot u, v \rangle &= \langle g^{-1} g u, g^{-1} v \rangle \\ &= \langle u, g^{-1} v \rangle \qquad g^{-1} v \in V \text{ since } V - G\text{-module} \\ &= 0 \quad \forall v \in V \end{aligned}$$

Hence $g \cdot u \in V^\perp$ $\therefore$ $V^\perp$ is submodule of G.

Proof continued

Let $n = \dim V$.
If $n = 1$ then V is irred.
Assume that every G module of $\dim < n$ is a direct sum of irred. Let V of $\dim n$.
If V is irred. then you are done. If V not it has a non zero proper submoduls.

$$V = S \oplus S^{\perp}$$
$$\dim S < \dim V$$
$$\dim S^{\perp} < \dim V.$$
//

remark: same Th$^{\underline{m}}$ holds for compact group acting on Hilbert space. Have to work with maximial orthonormal sets, analysis gets involved. $\mathcal{H} = \bigoplus_{\alpha} \mathcal{H}_{\alpha}$, can prove $\mathcal{H}_{\alpha}$ is finite dim'l, of course $\bigoplus_{\alpha}$ is over ctbly many of $\mathcal{H}_{\alpha}$.

remark: we have some notational options,

$$l(g) = l_g : V \longrightarrow V$$

the G-invariant inner product has

$$\langle l_g x, l_g y \rangle = \langle x, y \rangle \quad \forall x, y$$

$\therefore$ l_g is a unitary map

$\therefore$ a compact group rep. is necessarily unitary (inner-product prerserving)

$$g \longrightarrow \underset{\left(\substack{\text{relative to}\\ \text{orthonormal}\\ \text{basis}}\right)}{M(l_g)} \in U(N)$$

$\therefore$ rep. of compact G can be viewed as matrix-reps in ~~on~~ $U(N)$, $G \longmapsto U(N)$

SCHUR'S LEMMA

Th^m/ Let G be any group (Lie group, or topological, not necc. compact) and V and W irred. modules. Then

(i.) A morphism $f: V \to W$ is 0 or an isomorphism.

(ii) Every morphism $f: V \to V$ has the form $f = \lambda\, id_V$ for some $\lambda \in \mathbb{C}$.

(iii) $\dim_{\mathbb{C}} (Hom_G(V, W)) = 1$ if $V \cong W$

$\dim_{\mathbb{C}} (Hom_G(V, W)) = 0$ if $V \not\cong W$

Proof:

(i) Let $f: V \longrightarrow W$ be a equivariant G-map.

Note that $\ker f$ is a submodule.

$$f(v) = 0 \quad \Rightarrow f(gv) = gf(v) = 0$$

therefore $\ker f = 0$ or $\ker f = V$ since V is assumed to be irred. If $\ker f = V$ then $f = 0$ otherwise $\ker f = 0 \Rightarrow f$ injective

$f(V)$ is a submodule of W

thus $f(V) = W$ since W irred & $f(V) \neq 0$.

(ii.) If $f \equiv 0$ then $f = 0 \cdot id_V$. If $f \neq 0$ then we can look at matrix A_f and $\exists\, \lambda \in \mathbb{C}$ & $v \in V$ $v \neq 0$ s.t. $f(v) = \lambda v$. Consider

$Eigen_\lambda = \{ w \in V \mid f(w) = \lambda w \}$ eigen space of ~~λ~~.

$$f(gw) = g f(w) = g(\lambda w) = \lambda g w$$

$\therefore$ $Eigen_\lambda$ is submodule, not zero $\therefore = V$ since V is irred. $\therefore$ $f = \lambda\, id_V$

Proof cont'd

(iii) $\mathrm{Hom}_G(V, W) = 0$ if $V \not\cong W$ as G-modules

by (i)

When ~~$V \not\cong W$~~ $V \cong W$

$\mathrm{Hom}_G(V, V) = \{\lambda\, id_V \mid \lambda \in \mathbb{C}\} \Rightarrow \dim = 1.$

Remark:

$U(n) \longrightarrow Gl(n, \mathbb{C})$

$A \longmapsto A$

$l_g : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, $l_g \in \mathrm{Aut}(\mathbb{C}^n)$, $l_g(x) = gx$

this is the standard representation.

- see 1.11 in Book for standard def^n of trivial rep and so on...

Prop: If V and W are G-modules then the direct sum $V \oplus W$ is a G-module w.r.t.

$$g(v, w) = (gv, gw)$$

Proof can verify easily $(g_1 g_2) \cdot (v, w) = g_1 \cdot (g_2 \cdot (v, w))$

$e(v, w) = (v, w)$.

remark: if we have a rep $G \xrightarrow{A} Gl(n, \mathbb{C})$

and $G \xrightarrow{B} Gl(m, \mathbb{C})$ where $\dim V = n$, $\dim W = m$.

$$g \longrightarrow \begin{pmatrix} A(g) & 0 \\ 0 & B(g) \end{pmatrix}$$

$$G \longrightarrow Gl(m+n, \mathbb{C}).$$

just matrix rep of version of prop.

11/17/06.5

Prop. an irreducible representation of an Abelian Lie group G is one-dim'l.

Proof: Notice that if $g \in G$ then $l_g : V \longrightarrow V$ is a morphism.

$$l_g(hv) = (gh)v = hgv = h\, l_g(v)$$

but then according to Schur's Lemma.

$$l_g = \lambda(g)\, id_V$$

If S is nonzero subspace then it is invariant

$$l_g(S) = \lambda(g)\, id_V(S) \subset \lambda(g) S \subseteq S$$

hence every subspace is a submodule, but V is irred so it cannot have any subspaces except zero $\therefore$ V is one-dim'l. Also note

$$l_{gh} = \lambda(gh)\, id_V$$
$$l_g \circ l_h = \lambda(g)\lambda(h)\, id_V$$
$$\lambda : G \longrightarrow \mathbb{C}$$
$$\lambda(gh) = \lambda(g)\lambda(h)$$

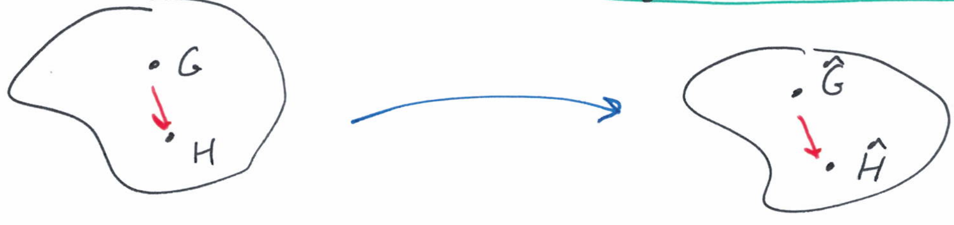
an Abelian Lie group G is one-one correspondance with homomorphisms of G into $\mathbb{C}$. In fact

$$\lambda : G \longrightarrow S^1$$

since λ unitary. Now Pontrajin theory

Locally Compact Groups

Dual, set of characters e.g. homomorphisms into S^1



(compact) $\longrightarrow$ (discrete) Fourier Series.

(discrete) $\longrightarrow$ (compact)

Prop. 1.14 next then onto characters. after linear alg.

Missed class before thanksgiving insert before this

(1) $\boxed{Hom(V, W) \cong V^* \otimes W}$

Define $\Theta : V^* \otimes W \longrightarrow Hom(V, W)$

$$\Theta(v^* \otimes w)(u) = v^*(u)w$$

Choose a basis $\{v_i\}$ of V and $\{w_j\}$ of W.

For $f \in Hom(V, W)$ we can calculate the matrix of f

$$f(v_i) = \sum_j r_{ji} w_j$$

then consider,

$$\Theta\Big(\sum_{i,k} r_{ik}(v_k^* \otimes w_i)\Big)(u) = \sum_{i,k} r_{ik}\, \Theta(v_k^* \otimes w_i)(u)$$

$$= \sum_{i,k} r_{ik}\, v_k^*(u)\, w_i$$

if we let $u = v_\ell$ we get

$$\Theta\Big(\sum_{i,k} r_{ik}\, v_k^* \otimes w_i\Big)(v_\ell) = \sum_{i,k} r_{ik}\, v_k^*(v_\ell)\, w_i$$

$$= \sum_{i,k} r_{ik}\, \delta_{k\ell}\, w_i$$

$$= \sum_{i,k} r_{i\ell}\, w_i = f(v_\ell)$$

Hence $\boxed{f = \Theta\Big(\sum_{i,k} r_{ik}\, v_k^* \otimes w_i\Big)}$

(2.) $\boxed{\begin{array}{l}\Theta : V^* \otimes W \longrightarrow \mathrm{Hom}_G(V, W) \\ \text{is a } G-\text{morphism.}\end{array}}$

$$\begin{aligned}\Theta\big(g\cdot(v^*\otimes w)\big)(u) &= \Theta\big((g\cdot v^*)\otimes(g\cdot w)\big)(u) \\ &= (g\cdot v^*)(u)\,(g\cdot w) \\ &= v^*(g^{-1}u)\,(g\cdot w) \\ &= g\cdot[v^*(g^{-1}u)\,w] \\ &= g\cdot\Theta(v^*\otimes w)(g^{-1}u) \\ &= [g\cdot\Theta(v^*\otimes w)](u)\end{aligned}$$

$v^*(g^{-1}u) \in \mathbb{C}$ and have linear group action,

Since G acts on $\mathrm{Hom}(V, W)$ via $(g\cdot f)(v) = g f(g^{-1}v)$ in the special case $W = \mathbb{C}$ $(g\cdot v^*)(u) = v^*(g^{-1}u)$.

Remark: the "G" on $\mathrm{Hom}_G(V, W)$ is possibly true, but we're not sure the range really is G-morphisms, this is seperate ? from Θ being G-morphism.

(3) For finite dim'l vector space V we have

$$\mathrm{Hom}(V, V) \cong V \otimes V$$

Define the trace mapping

$$Tr : V^* \otimes V \longrightarrow \mathbb{C}$$

$$Tr(v^* \otimes w) \equiv v^*(w)$$

this reduces to usual trace upon choosing basis. Define $Tr : \mathrm{Hom}(V, V) \longrightarrow \mathbb{C}$ by

$$Tr(f) = Tr(\Theta^{-1}(f))$$

where $\Theta : \mathrm{Hom}(V, V) \xrightarrow{\cong} V^* \otimes V$ from (1)

(3.) continued, lets check the "Tr" definitions match the usual idea of Trace on matrices.

It can be shown $\boxed{Tr\, f = \sum_i \lambda_{ii}}$ in terms of basis $\{V_i\}$ of V where $f(V_i) = \sum_j \lambda_{ji} V_j$

$$Tr(f) = Tr(\Theta^{-1} f)$$
$$= Tr\left(\sum_{i,k} \lambda_{ik}\, V_k^* \otimes W_i\right)$$
$$= \sum_i \lambda_{ii}$$

using other results from today.

• might look at prop. 3.3 in particular (i), (ii) and (v) maybe prove (i) then (iii) then (v).

Def^n/ The fixed point set of a representation V of a Lie group G is

$$V^G = \{ v \in V \mid g \cdot v = v \;\; \forall g \in G \}$$

Suppose $f : G \longrightarrow W$ is a function from G into a finite dim'l vector space (take home deals with $W = \mathbb{C}$) and that f is continuous with compact support then there is a unique vector which we denote

$$\int_G f(g)\, dg \in W$$

such that $\forall \alpha \in W^*$

$$\alpha\left(\int_G f(g)\, dg\right) = \int_G (\alpha \circ f)(g)\, dg$$

could write basis for W and work it out. Essentially just like for $f(x) = (f^1(x), f^2(x), \ldots, f^m(x))$

$$\int f(x)\, dx = \left(\int f^1(x)\, dx, \int f^2(x)\, dx, \ldots, \int f^n(x)\, dx\right)$$

from calc III. for integrating vector-valued fnct. If W is a Hilbert Space (also works for ∞-dim'l Hilbert Spaces) we would have

Continued

By Rieze Rep. Th^{m} every linear fnctnl on Hilbert space is an inner product so for $f: G \longrightarrow \mathcal{H}$

$$\boxed{\left\langle \int_G f(g)\,dg, w \right\rangle = \int_G \langle f(g), w\rangle\, dg}$$

in fact this works in Banach space as well provided $\exists$ sufficiently many linear fnctionals to seperate x from y that is $\exists\ \alpha$ s.t. $\alpha(x) \neq \alpha(y)$ when $x \neq y$.

Define $P: V \longrightarrow V$ by

$$P(v) = \int_G (g \cdot v)\, dg$$

for G – compact and $g \cdot v \in V$, continuous group action.

Remark: next show $P(V) \subset V^G$ (next time)

let $h \in G$ then

$$\begin{aligned} h \cdot P(v) &= h \cdot \int_G (g \cdot v)\, dg \\ &= \int h \cdot (g \cdot v)\, dg \\ &= \int (hg) \cdot v\, dg \\ &= \int (hg) \cdot v\, d(hg) \\ &= \int \bar{g} \cdot v\, d\bar{g} \\ &= P(v) \end{aligned}$$

sloppy left invariance

$\forall h \in G$ $\quad \therefore \underline{P(v) \in V^G}$

$P: V \longrightarrow V^G$ (it turns out)

Notice $P(V^G) = V^G$ can show its identity here. more next time.

If $f : G \longrightarrow V$ is continuous with compact support then

$$\boxed{\int_G f(g)\, dg \in V}$$

such that $\forall \alpha \in V^*$

$$\boxed{\alpha\left(\int_G f(g) dg\right) = \int_G (\alpha \circ f)(g) dg}$$

In case V has an inner product then

$$\boxed{\left\langle \int_G f(g) dg, w \right\rangle = \int_G \langle f(g), w \rangle dg}$$

for all $w \in V$. In this case $\alpha : V \to \mathbb{C}$ is simply just $\alpha(v) \equiv \langle v, w \rangle$. We also will have occasion to use $\alpha = \text{trace}(\,\cdot\,)$

//

Assume we have a compact group G with representation $P : V \longrightarrow V$

$$P(v) = \int_G (g \cdot v) dg$$

Claim: $p(v) \in V^G$ for every $v \in V$. Let $h \in G$ to $h \cdot P(v) = P(v)$

$$
\begin{aligned}
\langle h P(v), w \rangle &= \left\langle h \int_G (g \cdot v) dg, w \right\rangle \\
&= \left\langle \int_G (g \cdot v) dg, h^{-1} w \right\rangle \\
&= \int_G \langle g \cdot v, h^{-1} w \rangle dg \\
&= \int_G \langle hg \cdot v, w \rangle dg \\
&= \int_G \langle (hg) \cdot v, w \rangle d(hg) \quad \text{using "sloppy" left-invariance.} \\
&= \int_G \langle \bar{g} \cdot v, w \rangle d\bar{g} = \left\langle \int_G (\bar{g} \cdot v) d\bar{g}, w \right\rangle = \langle P(v), w \rangle
\end{aligned}
$$

Continuing the last calculation,

$$\langle h P(v) - P(v), w \rangle = 0$$

$\forall w \in V$ thus $hP(v) - P(v) = 0$ hence

$$\boxed{h \cdot P(v) = P(v)}$$

Next we show that P is a projection.

If $v \in V^G$

$$\begin{aligned}\langle P(v), w \rangle &= \left\langle \int_G (g \cdot v)\, dg, w \right\rangle \\ &= \int_G \langle g \cdot v, w \rangle\, dg \\ &= \int_G \langle v, w \rangle\, dg \\ &= \langle v, w \rangle \int_G dg = \langle v, w \rangle\end{aligned}$$

since

$\Rightarrow \langle P(v) - v, w \rangle = 0 \quad \forall w$

$\Rightarrow P(v) = v$

Remark: $\int_G f = \int_G f \Omega$

$\updownarrow$

$\int_G f(g)\, dg$

note $\int_G 1 = \int_G \Omega = \mu > 0$

let $\hat{\Omega} = \frac{1}{\mu} \Omega$ then $\int_G \hat{\Omega} = 1$.

Summary

$$\begin{aligned} h \cdot P(v) &= P(v) \quad \forall v \in V \\ P(v) &= v \quad \forall v \in V^G \\ P(v) &= \int_G (g \cdot v)\, dg \end{aligned}$$

$$V = V^G \oplus (V^G)^\perp$$

with respect to inner product $\langle , \rangle$.

Apply the projection P to $Hom(V,V)$

First, what is the fixed point set of $Hom(V,V)$

well G acts on $Hom(V,V)$ via

$$(g\cdot f)(v) = (g\cdot f)(g^{-1}v)$$

So $f \in Hom(V,V)^G$ iff

$$\begin{aligned} g\cdot f = f &\iff g f(g^{-1}v) = f(v) \quad \forall v\in V, \forall g \in G \\ &\iff f(g^{-1}v) = g^{-1}f(v) \quad \forall v \in V, \forall g \in G \\ &\iff f(hv) = h f(v) \quad \forall h \in G, v \in V \\ &\iff f \text{ is a morphism.} \end{aligned}$$

$$\boxed{\text{Proposition: } Hom(V,V)^G = Hom_G(V,V)}$$ (proof above)

Th^m/ Let V be an irrep. of a compact Lie group G then $\forall f \in Hom(V,V)$ (prop 4.2)

$$\int_G (g\cdot f)\,dg = \left[\frac{1}{\dim_{\mathbb{C}} V}\right] \text{Trace}(f)\ id_V.$$

Proof: For $f \in Hom(V,V)$ consider $P(f) = \int_G (g\cdot f)\,dg \in Hom_G(V,V)$ thus $P(f) : V \to V$ is a morphism, by Schur's Lemma

$$P(f) = \lambda_f\, id_V \quad \text{for some } \lambda_f \in \mathbb{C}$$

Notice then

$$\begin{aligned} (\dim_{\mathbb{C}} V)\lambda_f &= Tr(\lambda_f\, id_V) \\ &= Tr(P(f)) \\ &= Tr\left(\int_G (g\cdot f)\,dg\right) \\ &= \int_G Tr(g\cdot f)\,dg \\ &= \int_G Tr(g(f(g^{-1}(\cdot)))) \\ &= \int_G Tr(l_g \circ f \circ l_g^{-1})\,dg = \int_G Tr(f)\,dg = Tr f \cancel{\int_G dg}^{\,1} \end{aligned}$$

$$\therefore \boxed{\lambda_f = \frac{Tr f}{\dim_{\mathbb{C}} V}}$$

Notice that if $\{V_i\}$ is a basis of a rep. space V of a Lie group G we have a matrix rep.

$$g \longrightarrow (r_{ij}(g))$$

from G into $Gl(n, \mathbb{C})$ defined by

$$g \cdot V_j = \sum_i r_{ij}(g) V_i$$

you can check to see its a rep, ie a group homomophism. If $\{V_j^*\}$ is the basis of V^* dual to $\{V_i\}$ then we can hit both sides with a dual vector V_k^* to get $V_k^*(V_i) = \delta_{ki}$ resulting in,

$$\boxed{r_{ij}(g) = V_i^*(g \cdot V_j)}$$

this links the abstract linear op. theory to the explicit matrix group version of these things.

Defn/ A function $f: G \longrightarrow \mathbb{C}$ ~~from a~~ relative to a representation space V of a Lie group G into $\mathbb{C}$ is called a representative function iff $\exists\, \varphi \in V^*$, $v \in V$ such that

$$f(g) = \varphi(g \cdot v) \quad \forall g \in G$$

• notice that $r_{ij}: G \longrightarrow \mathbb{C}$ is a representative function.