Friday late 10 minutes. 12/1/06.1

Thm/ Let V be irrep of compact Lie group then for $f \in Hom(V, V)$

$$\int_G (g \cdot f)\, dg = \frac{1}{\dim_{\mathbb{C}} V} Tr(f)\, id_V$$ (prop 4.2)

Def^n/ If V is a rep of G

$f : G \to \mathbb{C}$ is a representative fnct iff

$\exists\, \varphi \in V^*$, $v \in V$ such that

$$f(g) = \varphi(gv)$$

$$r_{ij}(g) = V_j^*(g \cdot V_i)$$

defines such fncts.

Cor. V, G as before. for $\varphi \in V^*$, $v \in V$

$f \in Hom(V, V)$

(prop. 4.4) $$\int_G \varphi(g(f(g^{-1}v)))\, dg = \left(\frac{1}{\dim_{\mathbb{C}} V}\right)(Tr f)\, \varphi(v)$$

Let $v \in V$. fix $f \in Hom(V, V)$, $\beta = ev_v$

$$\beta : Hom(V, V) \longrightarrow V \xrightarrow{\alpha} \mathbb{C}$$

$\beta(h) = h(v)$

$$\beta\left(\int (g \cdot f)\, dg\right) = \int_G \beta(gf)\, dg$$

$$\alpha\left[\beta\left(\int_G (g \cdot f)\, dg\right)\right] = (\alpha \cdot \beta)\left(\int_G (g \cdot f)\, dg\right) = \int_G (\alpha \cdot \beta)(g \cdot f)\, dg$$

$$\alpha\left[\beta\left(\int_G (g\cdot f)\,dg\right)\right] = (\alpha\cdot\beta)\left(\int_G (g\cdot f)\,dg\right)$$

$$= \int_G (\alpha\cdot\beta)(g\cdot f)\,dg$$

$$= \int_G \alpha(\beta(g\cdot f))\,dg$$

$$= \alpha\left(\int_G \beta(g\cdot f)\,dg\right) \quad \forall\, \alpha \in V^*$$

$$\therefore \quad \beta\left(\int_G (g\cdot f)\,dg\right) = \int_G (g\cdot f)(v)\,dg$$

then use the Th^{m} that $\int_G (g\cdot f)\,dg = \frac{1}{\dim_{\mathbb{C}} V}(\mathrm{Tr}\, f)\, id_V$ to obtain that

$$\frac{1}{\dim_{\mathbb{C}} V}(\mathrm{Trace}\, f)\, id_V(v) = \int_G (g\cdot f)(v)\,dg$$

act on both sides by φ

$$\frac{1}{\dim_{\mathbb{C}} V}\,\mathrm{Trace}(f)\,\varphi(v) = \int_G \varphi((g\cdot f)(v))\,dg$$

$$= \int_G \varphi(g f(g^{-1}v))\,dg$$

Th^m/ Let V be an irred. rep. of a
(Th^m 45) compact Lie group G. For $v, w \in V$

(i.) $\int_G \langle g f(g^{-1} v), w \rangle \, dg = \frac{1}{\dim_{\mathbb{C}} V} (\mathrm{Tr} f) \langle v, w \rangle$

(ii) for $v, w, \alpha, \beta \in V$

$$\int_G \langle g^{-1} v, \alpha \rangle \langle g \beta, w \rangle \, dg = \frac{1}{\dim_{\mathbb{C}} V} \langle \beta, \alpha \rangle \langle v, w \rangle$$

Proof: Use corollary. Choose $\varphi \in V^*$, $\varphi(x) = \langle x, w \rangle$

$$\int_G \langle g f(g^{-1} v), w \rangle \, dg = \int_G \varphi(g f(g^{-1} v)) \, dg$$
$$= \frac{1}{\dim V} \mathrm{Tr}(f) \, \varphi(v)$$
$$= \frac{1}{\dim V} \mathrm{Tr}(f) \langle v, w \rangle \qquad \text{proves (i.)}$$

if give next one will give hint that should choose

$$f(x) = \langle x, \alpha \rangle \beta$$

then use # i with f specialized this way.

$$\int_G \langle g f(g^{-1} v), w \rangle \, dg = \int \langle g \langle g^{-1} v, \alpha \rangle \beta, w \rangle \, dg$$
$$= \frac{1}{|V|} \mathrm{Tr}(f) \langle v, w \rangle$$

Now $\langle g \langle g^{-1} v, \alpha \rangle \beta, w \rangle = \langle g^{-1} v, \alpha \rangle \langle g \beta, w \rangle$ so

$$\int_G \langle g^{-1} v, \alpha \rangle \langle g \beta, w \rangle \, dg = \frac{1}{|V|} \mathrm{Tr}(f) \langle v, w \rangle$$

now just need $\mathrm{Tr}(f) = \langle \beta, \alpha \rangle$ to finish

Proof continued

Let $\{V_i\}$ be an orthonormal basis

$$
\begin{aligned}
f(V_i) &= \langle V_i, \alpha \rangle \beta \\
&= \langle V_i, \sum_k \alpha^k V_k \rangle \sum_l \beta^l V_l \\
&= \sum_k \bar{\alpha}^k \langle V_i, V_k \rangle \sum_l \beta^l V_l \\
&= \sum_l \bar{\alpha}^i \beta^l V_l
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Trace}(f) &= \sum_i f(V_i) \cdot V_i \\
&= \sum_i \bar{\alpha}^i \beta^i \\
&= \langle \beta, \alpha \rangle
\end{aligned}
$$

Thm/ Let V irred. rep. of a compact Lie group. Then

$$\int_G \langle g^{-1}v, \alpha\rangle\langle g\beta, w\rangle dg = \frac{1}{\dim V}\langle\beta,\alpha\rangle\langle v, w\rangle$$

for $\alpha, \beta, v, w \in V$.

Thm/ Let V, W be nonisomorphic irred. rep of G and $\langle,\rangle_V$, $\langle,\rangle_W$ G-invariant inner products on $V+W$.

$$\int_G \overline{\langle g\alpha, v\rangle}_V \langle g\beta, w\rangle_W dg = 0$$

$\forall \alpha, v \in V$, $\beta, w \in W$

<u>Proof</u>: fix $\alpha \in V$ and $\beta \in W$ and define

$$b(v,w) = \int_G \overline{\langle g\alpha, v\rangle}\langle g\beta, w\rangle dg$$

Notice that b is conjugate linear in w and linear in v.
moreover b is G-invariant.

$$\begin{aligned} b(hv, hw) &= \int_G \overline{\langle g\alpha, hv\rangle}\langle g\beta, hw\rangle dg \\ &= \int_G \overline{\langle h^{-1}g\alpha, v\rangle}\langle h^{-1}g\beta, w\rangle\, d(h^{-1}g) \\ &= \int_G \langle \bar{g}\alpha, v\rangle\langle \bar{g}\beta, w\rangle d\bar{g} \\ &= b(v,w) \end{aligned}$$

Want to use Schurr's Lemma, convert to Hom,

$b : V \times \overline{W} \longrightarrow \mathbb{C}$

$\hat{b} : V \longrightarrow \overline{W}^*$ where $\hat{b}(v) \equiv b(v, \cdot)$

$\hat{b}(v)(w) \equiv b(v,w)$

need to show $\hat{b}$ is a morphism

$$\hat{b}(hv)(w) = b(hv, w) = b(v, h^{-1}w) = \hat{b}(v)(h^{-1}w) \equiv (h\cdot\hat{b})(w)$$

$\therefore\ \hat{b}(hv) = h\hat{b}(v)\ \forall v, \forall h \in G.$

this proves that

$$\hat{b} : V \longrightarrow \overline{W}^* \quad \text{is a morphism}$$

remark: book discusses $\overline{W}^*$ and the group action on it. In fact $\overline{W}^* \cong W$ (G-morphism) thus $\overline{W}^*$ is irreducible. Hence $\hat{b}$ is either zero or an isomorphism. If $\hat{b}$ were isomorphism

$$V \cong \overline{W}^* \cong W \quad \therefore \rightarrow\leftarrow \quad \text{since } V \not\cong W$$

$$\therefore \underline{\hat{b} = 0}$$

continuous fncts of compact support.

Define an inner product on $C^0(G, \mathbb{C})$ by

$$\langle \varphi, \psi \rangle = \int_G \varphi(g) \overline{\psi(g)}\, dg$$

$$L^2(G)$$

Let $v_1, v_2, \ldots, v_m$ and $w_1, w_2, \ldots, w_n$ be orthonormal bases of V and W which are irrep of a compact Lie group G, relative to invariant inner products the orthonormality is taken $\langle , \rangle_W$, $\langle , \rangle_V$

Define then the matrix rep from V,

$$(\Omega_{ij}^V)(g) = \langle g v_j, v_i \rangle_V$$

$$g \cdot v_j = \sum_i \Omega_{ij}^V(g)\, v_i$$

and same from W,

$$(\Omega_{ij}^W)(g) = \langle g w_j, w_i \rangle_W$$

$$~~(\Omega_{kl}^W)(g) = \sum_k \Omega~~$$

continuing, now applying the Th^{m}'s from 1. 12/4/06.3

$$\int_G r_{ij}^{V}(g)\,\overline{r_{kl}^{W}(g)}\,dg = \int_G \langle g v_j , v_i \rangle_V \overline{\langle g w_l , w_k \rangle_W}\,dg$$

$$= \overline{\int_G \overline{\langle g v_j , v_i \rangle_V} \langle g w_l , w_k \rangle_W \,dg}$$

$$= 0 \qquad (V \not\cong W)$$

$$\int_G r_{ij}^{V}(g)\,\overline{r_{kl}^{V}(g)}\,dg = \int_G \langle g v_j , v_i \rangle_V \overline{\langle g v_l , v_k \rangle_V}\,dg$$

$$= \int_G \langle g v_j , v_i \rangle_V \overline{\langle g^{-1} g v_l , g^{-1} v_k \rangle_V}\,dg$$

$$= \int_G \langle g v_j , v_i \rangle \langle g^{-1} v_k , v_l \rangle \,dg$$

$$= \cancel{\frac{1}{\dim V} \langle v_k , v_j \rangle \langle v_j , v_l}$$

$$= \int_G \langle g^{-1} v_k , v_l \rangle \langle g v_j , v_i \rangle \,dg$$

$$= \frac{1}{\dim V} \langle v_j , v_l \rangle \langle v_k , v_i \rangle$$

$$= \frac{1}{\dim V} \delta_{ik} \delta_{jl}$$

12/4/06.4

$\boxed{\begin{array}{l}\text{Def}^{\underline{n}}\text{/ Let } g \longrightarrow l_g \text{ be a representation of} \\ \text{a compact Lie group } G \text{ on } \operatorname{Aut}(V). \\ \text{The character of the representation is the fnct} \\ \qquad \chi_V : G \longrightarrow \mathbb{C} \text{ defined by} \\ \qquad \chi_V(g) = \operatorname{trace}(l_g)\end{array}}$

<u>Comments on final hwk</u>

p. 46. $l_g^* \alpha = \alpha$ left invariance.

$\alpha_e \in T_e^* G$, $\alpha_e : T_e G \longrightarrow \mathbb{R}$

↑ propagate over group via pullback.

basis of $T_e^* G$ wedged gives n-form on $\Lambda^n (T_e G)$

$$\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n \equiv \omega_e$$

then $\Omega_g = l_g^* \omega_e$ gives left invariant n-form on G

Now he defines how to integrate $f : G \to \mathbb{R}$ (later $\mathbb{C}$ $f = u + iv$ and such)

$$\int_G f = \underbrace{\int_G (f\Omega)}$$

↑ int. of n-form defined in book.

$$\int_G f(g)\, dg = \int_G f\Omega$$

he'll prove left-invariance.

$$\int_G 1\, dg = \int_G \Omega = \mu \neq 0 \quad \text{then}$$

$$\hat{\Omega} = \frac{1}{\mu}\Omega \text{ gives } \int_G 1\, dg = 1$$

Left-invariance of integral

$$\int_G f(hg)\,dg = \int_G (f\circ l_h)(g)\,dg$$

$$\equiv \int_G (f\circ l_h)\,\Omega$$

$$= \int_G (f\circ l_h)(l_h^*\Omega) \qquad (d(hg))$$

$$= \int_G l_h^*(f)\, l_h^*(\Omega)$$

$$= \int_G l_h^*(f\Omega) \qquad \text{by 5.8}$$

can assume $\varepsilon = 1$

$$= \int_G f\Omega$$

$$= \int_G f(g)\,dg$$

12/4/06. 6

$$\int_G (f \circ l_h \circ \varphi)\, dg = \int (f \circ \varphi)\, dg$$ (has to show this true for l_h, r_h, $inv = \varphi$

$$\Rightarrow \int_G f(g)\, dg = \int_G f\, dg = \int (f \circ \varphi)\, dg$$

One Th^m left. The one on pg. 81 ~~f~~ missed 12/6/06 Hannah Sezh 12/9/06. 1 12/8/06. actually

Th^m/ A representation space V (finite dim'l of compact Lie group) is determined by its character χ.

$Irr(G, \mathbb{C}) = \{$all finite dim'l irred rep of compact Lie group$\}$ / Isomorphism of vect. spaces

Proof: Recall that if V is irred. then

$V \cong W$ for some unique $W \in Irr(G, \mathbb{C})$.

A general finite dim'l rep. of G we know

$$V = \bigoplus_j n_j V_j$$

Where for each j, V_j is (an irred.) submodule of V and n_j is the multiplicity of V_j. Thus for $j \neq k$ $V_j \ncong V_k$. Its actually an orthogonal direct sum w.r.t. invariant inner-product

- This decomp is not unique however it is in $Irr(G, \mathbb{C})$ since we pick a representative of each $\cong$ class. So Moreover each V_j is isomorphic to one and only one $W_j \in Irr(G, \mathbb{C})$ and $n_j = \dim_{\mathbb{C}} Hom_G(V_j, V) = \dim_{\mathbb{C}} Hom_G(W_j, V)$

def^n a while back we showed this matches intuitive idea of multiplicity

$$V \cong \bigoplus_j n_j W_j$$

$$\langle \chi_V, \chi_{W_j} \rangle = \sum_j n_k \underbrace{\langle \chi_{W_k}, \chi_{W_j} \rangle}_{\delta_{kj}} = \sum_j n_k \delta_{kj} = n_j$$

Thus $W_j \in \mathrm{Irr}(G, \mathbb{C})$ is a summand in the decomposition of V iff

$$\langle \chi_{W_j}, \chi_{V_k} \rangle = 1$$

and its multiplicity is $n_j = \langle \chi_V, \chi_{W_j} \rangle$

- The W's are a particular choice so referring to them takes away from the ambiguity of the direct sum decomp. of the V_j's.

Proposition: If V is a representation and $\langle \chi_V, \chi_V \rangle = 1$ then V is irred.

Proof: Write $V = \bigoplus_j n_j V_j$

$$\langle \chi_V, \chi_V \rangle = \Big\langle {\sum_k}' n_k \chi_{V_k}, {\sum_l}' n_l \chi_{V_l} \Big\rangle$$

$$= {\sum_{k,l}}' n_k n_l \langle \chi_{V_k}, \chi_{V_l} \rangle$$

$$= \sum_{k,l} n_k n_l \delta_{kl} = \sum_j (n_j)^2 = 1$$

Thus as $n_j \geq 0 \Rightarrow n_j = 0 \ \forall j$ except one value of j say $j = j_0$ then $n_{j_0} = 1$ hence $V = V_{j_0}$ that is V is irred.

Prop: If V is an irred. rep. of a Lie group G which is compact and W is an irred. rep. of a compact Lie group H then $V \otimes W$ is an irred. rep of $G \times H$. Every irred. rep of $G \times H$ is of this form. (will not prove last sentence, but has it in notes if interested.) in book it doesn't detail actions it implicits.

Proof:

$$\langle \chi_{V \otimes W}, \chi_{V \otimes W} \rangle \stackrel{?}{=} 1$$

$$= \int_{G \times H} \chi_{V \otimes W}(g,h) \overline{\chi_{V \otimes W}(g,h)}\, dg\, dh$$

$$= \int_{G \times H} \chi_V(g) \chi_W(h) \overline{\chi_V(g) \chi_W(h)}\, dg\, dh$$

$\chi_{V \times W} = \chi_V \chi_W$

G rep. H rep.

$$= \int_{G \times H} \chi_V(g) \overline{\chi_V(g)}\, \chi_W(h) \overline{\chi_W(h)}\, dg\, dh$$

$$= \int_G \chi_V(g) \overline{\chi_V(g)}\, dg \int_H \chi_W(h) \overline{\chi_W(h)}\, dh$$

$$= \langle \chi_V, \chi_V \rangle \langle \chi_W, \chi_W \rangle$$

$$= (1)(1)$$

$$= (1).$$

12/9/06.4

G

$C^\circ(G, \mathbb{K})$

$$|f| = \sup_{x \in G} |f(x)|$$

$$\langle f, g \rangle = \int_G f(x)\, \overline{g(x)}\, dx$$

$L^2(G)$

$$\|f\| = \sqrt{\langle f, f \rangle}$$

G - compact

V - f. dim'l rep of G

$\{e_i\}$ - orthonormal basis of V

$\Omega_{ij} : G \longrightarrow \mathbb{K}$ ← we argued these are smooth before.

$$g \cdot e_i = \sum_j \Omega_{ji}(g)\, e_j$$

$$s(\alpha \otimes v)(g) = \alpha(g \cdot v) \quad \forall \alpha \in V^*, v \in V, g \in G$$

the output is in $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

this makes s a rep. fnct.

then on pg. 18 are the conclusions.

extend s linearly. Denote

① $S(V) \equiv \mathrm{im}(s) \subseteq C^\circ(G, \mathbb{C})$

$S(V)$ is spanned by $\{\Omega_{ij}\}$

↑ finite set of maps special rep. fncts.

(2) $\Omega_{ij}^{V} \perp \Lambda_{kl}^{W}$ for $V \not\equiv W$

$$S(V) \perp S(W)$$

Now take $L_g, R_g : C^0(G, \mathbb{K}) \longrightarrow C^0(G, \mathbb{K})$

$$(R_g f)(x) = f(xg), \quad x \in G$$
$$(L_g f)(x) = f(g^{-1}x), \quad x \in G$$

the reason for the L_g and R_g ~~use~~ is see (3)

$$R_g\left[s(\alpha \otimes v)\right] = s\left[\alpha \otimes (gv)\right]$$

R_g acts on $S(V)$, it forms a submodule under the R_g action

$$L_g\left[s(\alpha \otimes v)\right) = s\left((g\alpha) \otimes v\right)$$

Then he extends ~~rep~~. defn of rep. fnct. to the "real" defn (as opposed to baby version earlier) see pg. (19). A function $f \in C^0(G, \mathbb{K})$ is a representative fnct. if the following is finite dim'l,

$$G \underset{R}{\bullet} f \equiv \left\langle \{ R_g f \mid g \in G \} \right\rangle_{\text{span } \mathbb{K}}$$

Th^m on (20) 12/4/06. 6

if $G_{R}^{\circ} f$ is finite dim'l then likewise
$G_{L}^{\circ} f = \langle \{ L_g f \mid g \in G \} \rangle_{\mathbb{K}}$ is finite dim'l

$\mathcal{T}(G, \mathbb{K})$ = set of all rep. fncts.

$\mathcal{T}(G, \mathbb{K})$ forms linear space.

if $f, g \in \mathcal{T}(G, \mathbb{K}) \Rightarrow f \in S(V), g \in S(W)$
$\Rightarrow f+g \in S(V+W)$

same for product & conjugation. So you can show $\mathcal{T}(G, \mathbb{K})$ = direct sum of $S(V)$

pg. 129-130-131 (analysis needed for Peter Weyl Th^m)

$\mathcal{T}(G, \mathbb{K})$ dense in $C^0(G, \mathbb{K})$ sup norm
and dense in $L^2(G)$ (L^2 norm). This says matrix rep dense.
orthon. objects dense in Hilbert space

read p.g. 136-138 to see real Th^m.