# Lie Groups : FALL 2006 : Ha 265



Problems due on friday next.

$\boxed{\text{Hwk 1}}$ pg. 10 # 1 & 6 — gave handout to break it down a little.

He'll follow his notes for a while.

Def$^n$/ Let G be a group with a multiplicative group operation.
Let $m : G \times G \longrightarrow G$ be the product map

$$m(x, y) = xy$$

and let $i : G \longrightarrow G$ be the inversion map

$$i(x) = x^{-1}$$

G is a Lie group iff G is a $C^\infty$ manifold and m and i are $C^\infty$ maps.

remark: a $C^2$ group automatically has another atlas for which the group is $C^\infty$. So restricting to $C^\infty$ is not to restrictive, although it dodges some difficult th$^m$'s
See Van Wardenmm. QFT book.

(BD) Lie group is a smooth manifold which is also a group such that m is smooth

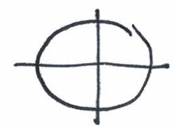
$$x^{-1}y = e$$

$$m(i(x), y) = e$$

Can use inverse fnct. theorem to prove i smooth near identity then can use left translations & Th$^m$ on pg. 126.

## Examples

(1.) $(\mathbb{R}^+, +)$

(2.) $U(1) = \{ z \in \mathbb{C} \mid |z| = 1 \}$ 

(3.) $Gl(n)$ = set of all nonsingular $n \times n$ real matrices.

Let $gl(n)$ denote the set of all $n \times n$ matrices

$$(A+B)^i_j = A^i_j + B^i_j$$

row

column

$$(\lambda A)^i_j = \lambda A^i_j$$

$$\| A \| = \sqrt{\sum_{i,j}^{n} (A^i_j)^2}$$

$$\begin{pmatrix} A^1_1 & A^1_2 \\ A^2_1 & A^2_2 \end{pmatrix} \overset{\psi}{\longmapsto} (A^1_1, A^1_2, A^2_1, A^2_2)$$

$\psi : gl(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$ and w.r.t. norm on $gl(n, \mathbb{R})$ just introduced $\|\psi(A)\| = \|A\|$. So $gl(n, \mathbb{R})$ is clearly a normed linear space (Banach Space)

The mapping

$$\tilde{m} : gl(n) \times gl(n) \longrightarrow gl(n)$$

$$\tilde{m}(A, B) = AB$$

is bilinear. Can calculate the 1st Frechet derivative on $gl(n) \times gl(n)$ easily, could give $gl(n) \times gl(n)$ a norm.

$$D_{(A,B)} \tilde{m}(H, K) = AK + HB$$

Alternatively $\tilde{m}^i_j(A,B) = (AB)^i_j = \sum_{k=1}^{n} A^i_k B^k_j$ = <u>polynomial</u>

$\therefore$ smooth.

(3.) Continued. If $A \in Gl(n)$ then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{cof}(A)$$

$$\operatorname{Cof}(A) = \text{cofactor matrix.}$$

Now then the f-la for $A^{-1}$ is a <u>rational</u> function of the entries of $A$ and as such it is smooth.

$$m = \tilde{m}\Big|_{Gl(n) \times Gl(n)}$$

$$Gl(n) = \det^{-1}((-\infty, 0) \cup (0, \infty)) = \{A \mid \det(A) \neq 0\}$$

Now $\det : gl(n) \longrightarrow \mathbb{R}$ is a continous fnct. and $\mathbb{R} - \{0\}$ is open therefore $Gl(n)$ is open in $gl(n)$. Hence $m = \tilde{m}/_{Gl \times Gl}$ is the restriction of the smooth map $m$ onto open subset of $gl(n)$ $\therefore$ it is smooth. $Gl(n)$ forms an open submanifold of $gl(n)$.

//

$$gl(n, \mathbb{C}) = n \times n \text{ complex matrices}$$

$$\boxed{\langle A, B \rangle = \sum_{i,j=1}^{n} A^i_j \, \bar{B}^i_j}$$

there is a similar real inner product on $gl(n, \mathbb{R})$. Anyway $gl(n, \mathbb{C})$ is a complex Hilbert space.

## Left Translation

$$G \times G \xrightarrow{m} G \quad \text{induces} \quad G \xrightarrow{l_a} G$$

$$\begin{array}{ccccc} G & \longrightarrow & \{a\} \times G & \xrightarrow{m} & G \\ x & \longrightarrow & (a, x) & \longrightarrow & ax \end{array}$$

$$l_a(x) = ax$$

$$l_a : G \to G$$

clearly smooth mapping on $G$.

Property : $l_a^{-1} = l_{a^{-1}}$.

Charts at identity give new charts under left translation $(U, x)_e \longmapsto (aU, x \circ l_a^{-1})$ and so the identity chart is important, the exponential map will cover that.

8/25/06

\#

~~Th$^{\underline{m}}$/ G a Lie Group with identity~~ e

Th$^m$/ G : Lie Group
e : identity
$(U, x)$ : chart of $G$ at $e$
Then
$l_a(U)$ is open in $G$ $\forall a \in G$

$$\mathcal{A} = \{ (l_a(U), x \circ l_a^{-1}) \mid a \in G \}$$

is a subatlas of the diff. structure of $G$

- A diff. structure is a <u>maximal</u> atlas. You can have different subatlases which will generate the same maximal atlas.
- $(V, y)$ a chart of some manifold then know $y(V) \overset{\text{open}}{\subseteq} \mathbb{R}^n$ and $V \xrightarrow{y} y(V)$ gives a diffeomorphism of $V$ & $y(V)$. So an alternative characterization of chart is that it is a diffeomorphism.
- $l_a^{-1} = l_{a^{-1}}$ note $l_a$ is diff. from $G$ to $G$ as discussed before. Thus

$$x \circ l_a^{-1} : aU \longrightarrow x(U)$$

is also a diffeomorphism $\therefore$ it is a chart, now is it admissable? If $x \circ l_a^{-1}$ & $x \circ l_b^{-1}$ are charts in $\mathcal{A}$ then $(x \circ l_a^{-1}) \circ (x \circ l_b^{-1})^{-1}$ is a diff.

$$(x \circ l_a^{-1}) \circ (x \circ l_b^{-1})^{-1} = x \circ l_a^{-1} \circ l_b \circ x^{-1}$$
$$= x \circ l_{a^{-1}b} \circ x^{-1}$$

Should worry about domains. When is it defined? when $aU \cap bU \neq \emptyset$ both charts defined on set, well more precisely $aU \cap bU$ has $x \circ l_a^{-1}$ & $x \circ l_b^{-1}$ well defined.
Now suppose $z \in aU \cap bU \Rightarrow z = au_1 = bu_2$
thus $b^{-1}a = u_2 u_1^{-1}$ don't know this in $U$ to begin.
So we should shrink $U$

wandering proof continued, $z = a u_1 = b u_2$

$$(x \circ l_a^{-1}) \circ (x \circ l_b)^{-1} = x \circ l_a^{-1} \circ l_b \circ x^{-1}$$
$$= x \circ l_{a^{-1}b} \circ x^{-1}$$

we'll need $a^{-1}b\, U \subseteq U$ to make sense.

$$\not\longrightarrow$$

remedy for proof.

Let $V$ contain $e$ such that $V^2 = VV \subseteq V$

we can do this since

$$m : G \times G \longrightarrow G$$
$$(e, e) \qquad \rotatebox{90}{$\subseteq$}\; V$$
$$\uparrow \quad \uparrow$$
$V_1$ $V_2$ both open around $e$ take $V_1 \cap V_2$

one step further,

$e \in W$ $\quad W^{-1} W \subseteq V$

$x^{-1}y$ continous map

probably should modify Th^m & work with $W$-type chart about the ~~ind~~ identity.

$$(x_W \circ l_a^{-1}) \circ (x_W \circ l_b^{-1}) = x_W \circ l_{a^{-1}} \circ l_b \circ x_W^{-1}$$
$$= x_W \circ l_{a^{-1}b} \circ x_W^{-1}$$

Then $(x \circ l_a^{-1}) \circ (x_W \circ l_b^{-1})^{-1} (x(W)) = x_W \circ l_{a^{-1}b}(W)$

$$= x_W(a^{-1}b\, W)$$
$$= x_W(w_1 w_2^{-1} W)$$

$a w_1 = b w_2$

$a^{-1}b = w_1 w_2^{-1}$

Long story short: need to shrink $U$ a little to insure compatibility.

<u>Remark</u>: if we already have a Lie Group then we have an atlas $\mathcal{A}$ as in Th$^m$. Alternatively if we have a group and we want to make it a Lie group we can proceed to construct it by finding chart at $e$ then left translate to build atlas.

$V$ : finite dim'l vector space

$g$ : metric
bilinear
symmetric
nondegenerate $\quad g(v,w) = 0 \ \forall w \Rightarrow v = 0$

If $\{e_i\}$ is a basis for $V$ then $g(e_i, e_j) = G_{ij}$ is an invertible matrix, this is an alternative def$^{\underline{n}}$ of non deg.

Since $g$ is symmetric there exists a basis such that $G_{ij} = \pm \delta_{ij}$. Then using $\vec{x}$ for components of $x$,

$$g(x,y) = \vec{x}^{\,t} G \vec{y}$$

If $\alpha$ is an isometry of $(V, g)$ then its matrix $A$ satisfies

$$\begin{aligned} (A\vec{x})^t G (A\vec{y}) &= g(\alpha(x), \alpha(y)) \\ &= g(x,y) \\ &= \vec{x}^{\,t} G \vec{y}. \quad \forall x, y \end{aligned}$$

Thus $\boxed{A^t G A = G} \iff g(\alpha(x), \alpha(y)) = g(x,y)$.

- Could do for complex #'s to, but need to work with complex metric.

Th$^{\underline{m}}$/ If $J \in gl(n)$ is such that $J^2 = I$ & $J^t = J$ then

$$G_J = \{ A \in gl(n) \mid A^t J A = J \}$$

is a Lie group. Also

$$G_J^{\mathbb{C}} = \{ A \in gl(n, \mathbb{C}) \mid A^\dagger J A = J \}$$

is also a real Lie group. Here $A^\dagger = \text{conj}(\text{transpose}(A))$

<u>remark</u>: book also does $J^2 = -I$ (symplectic)

<u>Proof</u>: $A \in G_J \Rightarrow \det(A^t)\det(J)\det(A) = \det(J)$

$\Rightarrow (\det(A))^2 = 1$ since $\det(J)^2 = 1$

$\Rightarrow \det(A) = \pm 1$

Thus $G_J \subset Gl(n)$ and we can show it's a group. The manifold structure can be defined several ways, level surface arguments or matrix exp. ideas.

<u>Lemma</u>: If $S$ is a submanifold of a manifold $M$ and $f : M \longrightarrow N$ is smooth map into manifold $N$ then $f|_S$ is smooth.

<u>Proof</u>: better to prove $i : S \hookrightarrow M$ is smooth. Then $f|_S \equiv f \circ i$ since $\Rightarrow f|_S$ smooth if $i$ smooth.

recall def$^{\underline{n}}$ of submanifold is, let $(U, x)$ be chart of $M$

$$x(U \cap S) = x(U) \cap \mathbb{R}^k \times \{0\}$$

$$x(U) \subseteq \mathbb{R}^m = \mathbb{R}^k \times \mathbb{R}^l$$

Then $S$ is a submanifold if it can be covered by submanifold charts.

$$x_s : U \cap S \longrightarrow x(U \cap S) \subseteq \mathbb{R}^k \quad \left( \text{identifying } \mathbb{R}^k \times \{0\} \equiv \mathbb{R}^k,\ (u) \longrightarrow (u, 0). \right)$$

need to show $x \circ i \circ x_s^{-1} : x(U \cap S) \subseteq \mathbb{R}^k \rightarrow x(U) \subseteq \mathbb{R}^m$ smooth

Last time correct but some comments to clarify perhaps

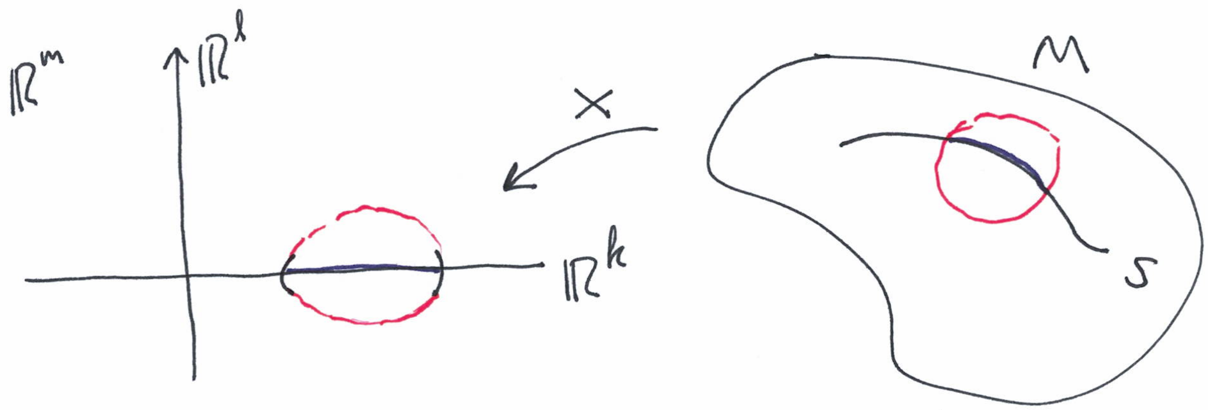
$$G_J = \{A \in gl(n) \mid A^t J A = J\}$$
$$G_J^{\mathbb{C}} = \{A \in gl(n,\mathbb{C}) \mid A^t J A = J\}$$

Then $G_J$ & $G_J^{\mathbb{C}}$ are real Lie Groups.

<u>Submanifold Map</u>

$S \subseteq M$ then $(U, x)$ has <u>submanifold</u> property if $U \cap S \neq \emptyset$ & $x(U \cap S) = x(U) \cap (\mathbb{R}^k \times \{0\})$ and $x(U) \overset{\text{open}}{\subseteq} \mathbb{R}^m = \mathbb{R}^k \times \mathbb{R}^l$

$$x_s : U \cap S \longrightarrow \mathbb{R}^k$$
$$x_s(w) = j(x(w))$$
$$j : \mathbb{R}^k \times \mathbb{R}^l \longrightarrow \mathbb{R}^k$$
$$j\big|_{\mathbb{R}^k \times \{0\}} : \mathbb{R}^k \times \{0\} \longrightarrow \mathbb{R}^k$$




it can be shown that S paired with the family of submanifold charts is a manifold.

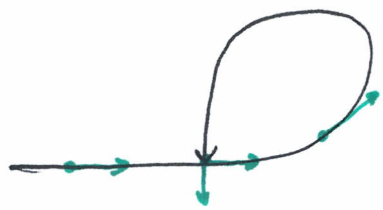
Let us defined an immersed submanifold.

$S$ a manifold and $i : S \longrightarrow M$ has

(1.) $i$ injective

(2.) $d_u i : T_u S \longrightarrow T_{i(u)} M$ injective. (?)

Then $i(S)$ is an immersed submanifold of $M$.



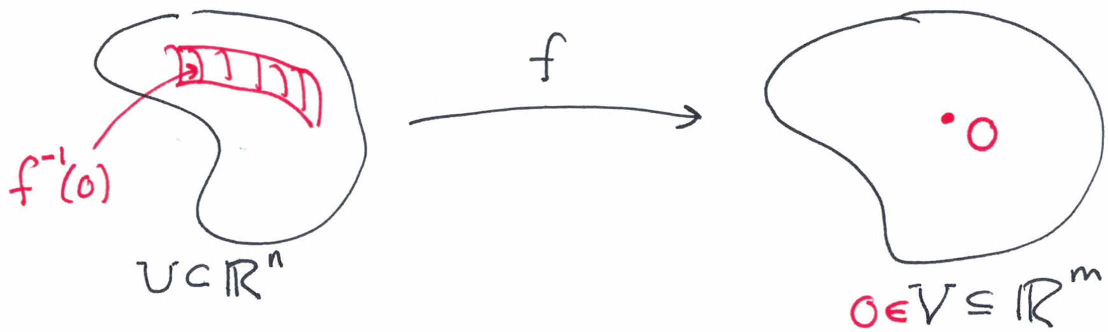
- this is an immersed submanifold BUT its not a ~~manifo~~ submanifold in the sense of finding family of submanifold maps.

The lemma last time could be stated

**Lemma**: Every ~~[illegible]~~ submanifold is an immersed submanifold. (But not the converse.)

Proof of $Th^m$

We show that $G_J$ is a submanifold of $gl(n)$. Recall that if $f: \mathbb{R}^n \to \mathbb{R}^m$ is smooth such that the Frechet derivative $D_p f$ has rank $m$ $\forall p \in f^{-1}(0)$ and if $n > m$ then $f^{-1}(0)$ is a submanifold of $\mathbb{R}^n$ of dimension $n-m$




$\dim(\mathrm{Ker}(D_p f)) = n-m.$

Observe that $gl(n) \cong \mathbb{R}^{n^2}$ and consider $f: gl(n) \to gl(n)$

$$f(A) = A^t J A - J$$

Then $G_J = f^{-1}(0) = \{ A \mid A^t J A - J = 0 \}$.

Notice we need the codomain of $f$ to have smaller dimension than domain, we can show that $\mathrm{Im}(f)$ lies on lower dim'l subset of $gl(n)$.

$$f(A)^t = (A^t J A - J)^t = A^t J A - J = f(A)$$

Thus $f(A)$ is symmetric. $f: gl(n) \to \mathcal{S}(n)$ where $\mathcal{S}(n) \equiv \{ B \in gl(n) \mid B^T = B \}$, $\dim(\mathcal{S}(n)) = \frac{1}{2} n(n+1)$

$f^{-1}(0) = G_J$ and $0 \in \mathcal{S}(n)$ (redefine codomain)

need to calculate the rank of the differential.

8/28/06.

Continuing to work on rank$(D_p f)$

$$f(A+H) = (A+H)^t J(A+H) - J$$
$$= A^t J A + H^t J A + A^t J H + H^t J H - J$$
$$= f(A) + \underline{H^t J A + A^t J H} + H^t J H$$
$$= f(A) + \underbrace{(D_A f)}(H) + H^t J H$$

defining $(D_A f)(H)$. we should show this is linear and satisfies correct limit is correct.

- we know it's smooth, we're just interested in the rank here. (once we prove that $D_A f$ is what it is)

$$0 \leq \frac{\| f(A+H) - f(A) - D_A f(H) \|}{\|H\|} \leq \frac{\| H^t J H \|}{\|H\|}$$
$$\leq \frac{\|H^t\| \|J\| \|H\|}{\|H\|}$$
gl(n) is Banach algebra.
$$\leq \|H^t\| \|J\|.$$

Now as $\|H\| \longrightarrow 0$ it follows that $\|H^t\| \longrightarrow 0$ hence the desired limit is obtained and we see that $(D_A f)(H)$ is indeed the derivative as claimed.

## Continuing to work on rank $(D_p f)$

$$(D_A f)(H) = A^t J H + A^t J H^t$$

Thus $(D_A f)(H)^t = (D_A f)(H)$ the differential is symmetric. Hence we observe for $A \in f^{-1}(0)$.

$$(D_A f)(H) = 0 \iff B = (A^t J H)^t = -(A^t J H)$$
$$\iff H = (A^t J)^{-1} B \text{ for } B^T = -B.$$
$$\iff \mathrm{Ker}(D_A f) \cong \{B \in gl(n) \mid B^T = -B\}$$

The mapping $L$

$$X \xmapsto{L} (A^t J)^{-1} X$$

is a linear map and in fact a vector space isomorphism (with inverse $Y \longmapsto A^t J Y$). Note that

$$L(\text{skew sym}) = \mathrm{Ker}(D_A f)$$

then this says that $A\mathcal{S}(n) \equiv \{B \mid B^T = -B\} \subset gl(n)$

$$\dim(\mathrm{Ker}\, D_A f) = \dim(A\mathcal{S}(n))$$
$$= \tfrac{1}{2} n(n-1).$$

Now $gl(n) = \mathcal{S}(n) \oplus A\mathcal{S}(n)$ since $A = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t)$.

so $\dim(A\mathcal{S}(n)) = \dim(gl(n)) - \dim(\mathcal{S}(n)) = n^2 - \frac{1}{2}n(n+1) = \frac{1}{2}n(n-1)$.

finally we know $rk(D_A f) + \nu(D_A f) = n^2 = \dim(\mathrm{dom}(D_A f))$

$$rk(D_A f) = n^2 - \nu(D_A f)$$
$$= n^2 - \tfrac{1}{2}n(n-1)$$
$$= \tfrac{1}{2}n(n+1).$$

$\boxed{\therefore f^{-1}(0) \text{ is a submanifold}}$

Thus $f : gl(n) \to \mathcal{S}(n)$ then $rk(D_A f)$ has rank $(D_A f) = \dim(\mathcal{S}(n))$.

We know submanifolds are immersed submanifolds thus as we have proven that $f^{-1}(0)$ is a submanifold

$$i : G_J \hookrightarrow Gl(n) \hookrightarrow gl(n)$$

$$\left.\begin{aligned} m &: Gl(n) \times Gl(n) \longrightarrow Gl(n) \\ inv &: Gl(n) \longrightarrow Gl(n) \end{aligned}\right\} \text{ know are smooth.}$$

$$m \circ (i \times i) : G_J \times G_J \xhookrightarrow{i \times i} Gl(n) \times Gl(n) \xrightarrow{m} Gl(n)$$

the lemma tells us $i \times i$ is smooth.

$$\operatorname{Im}(m \circ (i \times i)) \subseteq G_J \equiv G_J$$

Then finally the operations

$i \circ m \circ (i \times i)$

$i \circ inv \circ i$

are smooth on $G_J$.

## Comments on the Homework

$$\varphi(z) = \begin{bmatrix} \operatorname{Re} z & -\operatorname{Im} z \\ \operatorname{Im} z & \operatorname{Re} z \end{bmatrix}$$

$$\|\varphi(z)\|^2 = 2|z|^2$$

$$\|\psi(A)\|^2 = 2\|A\|^2 \Rightarrow \psi \text{ continuous map.}$$

$$\varphi(\bar{z}) = \varphi(z)^t$$

But it is not surjective $\psi : gl(n, \mathbb{C}) \longrightarrow gl(2n, \mathbb{R})$ and $\dim(gl(n, \mathbb{C})) = 2n^2$ whereas $\dim(gl(2n, \mathbb{R})) = 4n^2$.

- Continuous map of connected = connected.
  $\Rightarrow \psi(U(n)) \subset O(2n)$ must actually be $SO(2n)$ since the image is connected.

- Later $Gl(n, \mathbb{C}) \cong U(n) \times \mathbb{R}^{n^2}$

$$Gl^+(n, \mathbb{R}) = \{A \mid \det A > 0\} = \det^{-1}(\mathbb{R}^+)$$

$$Gl^-(n, \mathbb{R}) = \{A \mid \det A < 0\} = \det^{-1}(\mathbb{R}^-)$$

## Problem One

Smooth near identity then can prove smooth elsewhere via $l_a$ & $r_a$

$$f(A) = A^t J A - J$$

$$f^{-1}(0) = G_J$$

Similarly $G_J^{\mathbb{C}}$ stems from $f(A) = A^\dagger J A - A$

and $\mathcal{S}(n) \longrightarrow \mathcal{H}(n) = \{A \mid A^\dagger = A\}$ $\qquad B^\dagger = (\bar{B})^t$

$a\mathcal{S}(n) \longrightarrow a\mathcal{H}(n) = \{B \mid B^\dagger = -B\}$

and just as before $gl(n, \mathbb{C}) = \mathcal{H}(n) \oplus a\mathcal{H}(n)$.

$$\dim(\mathcal{H}(n)) = \frac{2n^2 - 2n}{2} + n = \cancel{2}n^2$$

real dimension

$$\Rightarrow \dim_{\mathbb{R}}(G_J^{\mathbb{C}}) = n^2$$

Since $f : gl(n, \mathbb{C}) \longrightarrow \mathcal{H}(n)$

$$\begin{aligned} \dim(f^{-1}(0)) = \dim(G_J^{\mathbb{C}}) &= \dim(\mathrm{dom}(f)) - \dim(\mathcal{H}(n)) \\ &= 2n^2 - n^2 \\ &= n^2 \quad \therefore \boxed{\dim(G_J^{\mathbb{C}}) = n^2} \end{aligned}$$

(get Morton Curtis' Lie groups back) 8/30/06.3

$A \in gl(n)$ the matrix exponential

$$\exp(A) = e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

$$A^0 = I$$

- Show its convergent sum

$$S_n = \sum_{k=0}^{n} \frac{1}{k!} A^k$$

$m > n$

$$\| S_m - S_n \| = \left\| \sum_{k=n+1}^{m} \frac{1}{k!} A^k \right\|$$

$$\leq \sum_{k=n+1}^{m} \frac{1}{k!} \|A\|^k$$

Let $s_n = \sum_{k=0}^{n} \frac{1}{k!} \|A\|^k = \cancel{e^{\|A\|}} < \infty$

then $0 \leq \| S_m - S_n \| \leq | s_m - s_n |$

now $| s_m - s_n | \to 0$ since $\lim_{n \to \infty} s_n = e^{\|A\|}$.

$\therefore \| S_m - S_n \| \longrightarrow 0$ $\therefore$

$\{S_m\}_{m=1}^{\infty}$ is Cauchy seq. in $gl(n)$

a Banach Space $\therefore$ $\{S_m\}_{m=1}^{\infty}$ converges

and we define $e^A$ to be the limit

of the sequence.

# Frechet Derivative of Exponential

$$D_A(\exp)(H) = H + \tfrac{1}{2}(AH + HA) + \tfrac{1}{6}(A^2H + AHA + HA^2) + \cdots$$

Proof:

$$\exp(A+H) - \exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[(A+H)^k - A^k\right]$$

$$= \sum_{k=1}^{\infty} \frac{1}{k!}\left[(A+H)^k - A^k\right]$$

$$= A + H - A + \sum_{k=2}^{\infty} \frac{1}{k!}\left[(A+H)^k - A^k\right]$$

$$= H + (A+H)^2 - A^2 + \sum_{k=3}^{\infty} \frac{1}{k!}\left[(A+H)^k - A^k\right]$$

$$= H + AH + HA + (A+H)^3 - A^3 + \cdots$$

Let $P(A^a, H^b)$ = all possible sums of terms with $a$ copies of $A$ & $b$ copies of $H$.

$$P(A^1, H^1) = AH + HA$$
$$P(A^0, H^2) = H^2$$

Notice then

$$(A+H)^3 = A^3 + \underbrace{\overbrace{A^2H + AHA + HA^2}^{k=3,\ i=1}}_{P(A^2, H)} + \underbrace{\overbrace{AH^2 + HAH + H^2A}^{k=3\ \ i=2}}_{P(A, H^2)} + H^3$$

$$\cancel{\exp(A+H) - \exp(A) = H + \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{i=1}^{k} P(A^{k-i}, H^i)}$$

~~notes wrong~~

note wrong..

8/30/06.5

Continuing

$$\exp(A+H) - \exp(A) = H + \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{i=1}^{k} P(A^{k-i}, H^{i})$$

If $A \neq 0$ and $\|H\| \leq \|A\|$

$$\left\| \exp(A+H) - \exp(A) - \left[ H + \sum_{k=2}^{\infty} \frac{1}{k!} P(A^{k-1}, H) \right] \right\|$$

$$= \cancel{\Big\| \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{i=1}^{k-1} P(A}$$

$$= \left\| \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{i=2}^{k} P(A^{k-i}, H^{i}) \right\|$$

$$\leq \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{i=2} \binom{k}{i} \|A\|^{k-i} \|H\|^{i}$$

$$\leq \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{i=2}^{k} \binom{k}{i} \|A\|^{k}$$

used $\|H\| < \|A\|$ to greedy

$$\leq \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{i=2} \binom{k}{i} \|A\|^{k-2} \|H\|^{2}$$

$$\leq \sum_{k=2}^{\infty} \frac{2^{k}}{k!} \|A\|^{k-2} \|H\|^{2}$$

then divide by $\|H\|$ to see that the diff. quotient goes to zero