Page 10 # 3, 4, 5 Due Friday 9/11/06.1

Hint # 5 D discrete subgroup of a Lie group G

Choose an open set U about $e \in G$ such that

$$U \cap D = \{e\}$$

(1.) Show $d \in D \Rightarrow dU \cap D = \{d\}$

(2.) Choose V open & connected about e such that

$$V^{-1} d V \subseteq dU$$

can do this by continuity of operations (show this)

(3) Prove with the choices above

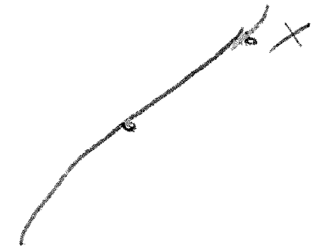
$$dx = xd \quad \forall x \in V$$

(4.) use previous problem can show

1.) Let $d \in D$ let $x \in G$ then $\exists \alpha : I \to G$

such that $\alpha(0) = e$ & $\alpha(1) = x$

$$\sigma_\alpha(t) \equiv \alpha(t)^{-1} d\, \alpha(t)$$



Can show

$\sigma_\alpha(t)$ is constant.

$$\sigma_\alpha(t) = \sigma_\alpha(0) = d.$$

then $d = \alpha(t^{-1})\, d\, \alpha(t) \quad \forall t$

$t = 1 \Longrightarrow d = x^{-1} dx$

$\Longrightarrow xd = dx$

Def^n/ A vector field $\mathfrak{X}$ on a Lie group G is a left invariant (right invariant) iff $\forall x \in G$, $\forall a \in G$,

$$d_x l_a(\mathfrak{X}_x) = \mathfrak{X}_{ax}$$

$$d_x l_a(\mathfrak{X}(x)) = \mathfrak{X}(l_a(x)) \quad (d_x r_a(\mathfrak{X}_x) = \mathfrak{X}_{xa})$$

Recall that

$v \in T_e G$

$$\mathfrak{X}^v(x) = d_e l_x(v)$$

$$\begin{aligned} d_x l_a(\mathfrak{X}^v(x)) &= d_x l_a(d_e l_x(v)) \\ &= d_e(l_a \circ l_x)(v) \\ &= d_e(l_{ax})(v) \\ &= \mathfrak{X}^v(ax) \end{aligned}$$

in fact this is the only way to get $\overbrace{\text{LIVF on } G}^{\Gamma \,///}$,

Th^m/ The mapping $\Phi : T_e G \longrightarrow \Gamma_{inv}(G)$ defined by

$$\Phi(v) = \mathfrak{X}^v$$

is a vector space isomorphism.

Remark: vector fields in $\Gamma_{inv}(G)$ are globally defined, so $\Gamma_{inv}(G)$ is a vector space more over it has ..

Proof: $v, w \in T_e G$

$$\Phi(v+w) = \mathfrak{X}^{v+w}$$

$$\mathfrak{X}^{v+w}(x) = d_e l_x(v+w) = d_e l_x(v) + d_e l_x(w) = \mathfrak{X}^v(x) + \mathfrak{X}^w(x) = (\mathfrak{X}^v + \mathfrak{X}^w)(x)$$

$$\Rightarrow \Phi(v+w) = \mathfrak{X}^{v+w} = \mathfrak{X}^v + \mathfrak{X}^w = \Phi(v) + \Phi(w).$$

similarly $\Phi(cv) = \mathfrak{X}^{cv} = c\,\mathfrak{X}^v = c\,\Phi(v)$

We prove Φ is injective.

Let $v \in \operatorname{Ker} \Phi \Rightarrow \Phi(v) = 0$

$$\Rightarrow X^v = 0$$

$$\Rightarrow X^v(e) = 0$$

$$\Rightarrow d_e l_e(v) = 0$$

$$\Rightarrow v = 0 \qquad \therefore \operatorname{Ker} \Phi = 0$$

$\therefore$ Φ injective.

To see surjectivity, Let X be a LIVF then let $v = X(e)$ and show $X = X^v$.

$$\begin{aligned} X(x) &= X(xe) \\ &= d_e l_x(X(e)) \\ &= d_e l_x(v) \\ &= X^v(x) \qquad \therefore \quad X = X^v \\ &\forall x. \end{aligned}$$

Remark: $\Gamma(M) = \Gamma_M$ denotes space of all vector fields on M (which are smooth) We know Γ is a vector space. We know Γ is a $C^\infty(M)$ - module

$$f \in C^\infty(M),\ X \in \Gamma \Rightarrow (fX)(x) = f(x)X(x) \in T_x M$$

$$\Rightarrow fX \in \Gamma$$

you can check the module properties. The LIVF are a finite dim'l subspace $\Gamma_{inv} \subset \Gamma$ but they're not a submodule

Remark continued

$\Gamma_{inv} \subseteq \Gamma$

$d_x l_a (f \mathfrak{X})$

$$\mathfrak{X} = \sum a^i \frac{\partial}{\partial x^i}$$

(locally) freely finitely generated as a module. (a^i fncts.)

$$\mathfrak{X}^v = \mathfrak{X}^{\sum_i v^i e_i} \quad : \text{where } \langle e_i \rangle = T_e G$$

$$= \sum_i v^i \mathfrak{X}^{e_i}$$

: where $\mathfrak{X}^{e_i}$ is seen to be the basis of the LIVF, notice v^i are #'s.

$\boxed{\text{Th}^m/ \text{ If } G \text{ is a Lie group then } TG \longrightarrow G \text{ is a trivial vector bundle.}}$

$$\begin{tikzcd} G \times T_e G \arrow[rr, "\varphi"] \arrow[dr, "\pi_G"'] & & TG \arrow[dl, "\tau_G"] \\ & G & \end{tikzcd}$$

meaning $\pi^{-1}(g) = \{g\} \times T_e G \quad \forall g \in G$.
also $\tau_G^{-1}(x) = T_x G$ and φ is a vector bundle isomorphism, it makes the diagram commute

Generally:

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d] & F \arrow[d] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

φ, f diffeomorphisms so that $\varphi(E_x) \subseteq F_{f(x)}$

Proof TG is trivial:

In contrast to $TS^2 \not\cong S^2 \times \mathbb{R}^2$, you can prove that for the tangent bundle to be trivial for M then if $n = \dim(M) > 0$ then its TM trivial $\exists$ n-global vector fields on M which are LI on the sphere can find 3 LI sphere that 2 of which span $T_p S^2$ but not the same two over all S^2.

Proof: $\varphi(g, v) \equiv (g, d_e l_g(v)) = (g, X^v(g)) \in T_g G$
this is smooth as fnct. of two variables (left trans)

$$\varphi(g_1, v_1) = \varphi(g_2, v_2) \Rightarrow (g_1, d_e l_{g_1}(v_1)) = (g_2, d_e l_{g_2}(v_2))$$

$$\Rightarrow g_1 = g_2 = g \;\&\; d_e l_g(v_1) = d_e l_g(v_2)$$

$$\Rightarrow g_1 = g_2 \;\&\; v_1 = v_2 \quad \text{by } d_e l_{g^{-1}} \text{ as } l_g \circ l_{g^{-1}} = e.$$

$$\Rightarrow \varphi \text{ one-one.}$$

To show onto its nearly the same argument.

G Lie Group

$\boxed{\text{TRIVIALITY OF } TG}$

Hwk pg. 22 # 3, 5, 8

$$\boxed{\begin{tikzcd} G \times T_e G \arrow[rr, "\varphi"] \arrow[dr, "\pi_G"'] & & TG \arrow[dl, "\tau_G"] \\ & G & \end{tikzcd}}$$

$$\varphi(g,v) = (g, d_e l_g(v)) \in TG, \quad d_e l_g(v) \in T_g G$$

To see that φ surjective choose $(h, w) \in TG$ then $w \in T_h G$ and $\therefore d_h l_{h^{-1}}(w) \in T_e G$.

$$\begin{aligned} \varphi(h, d_h l_{h^{-1}}(w)) &= (h, d_e l_h(d_h l_{h^{-1}}(w))) \\ &= (h, d_h(l_h \circ l_{h^{-1}})(w)) \\ &= (h, w) \qquad \therefore \varphi \text{ onto } TG \end{aligned}$$

- Just wanted to show that TG is trivial if G is a Lie Group.

Remark: If M is a manifold and X is a vector field on M then $\forall x \in M$, $X(x) \in T_x M$. This defines a section of the tangent bundle $TM \xrightarrow{\pi} M$

$$\tilde{X} : M \longrightarrow TM$$

$$\tilde{X}(x), (x, X(x))$$

others use notation $TM = \bigcup_{x \in M} T_x M$ but he needs the explicit pair for the sake of ~~spe~~ emphasizing coordinates vs velocities or momenta for T^*M. Dr. Fulp follows Marsden.

$$\pi(\tilde{X}(x)) = \pi(x, X(x)) = x$$

$$\pi \circ \tilde{X} = id_M \qquad (\tilde{X} \text{ is a section})$$

Continuing remarks on triviality of TG 9/13/06.2

$$\begin{tikzcd} G \times T_eG \arrow[r, "\varphi"] \arrow[d] & TG \arrow[d] \\ G & G \end{tikzcd}$$

$v \in T_eG$ $\qquad\qquad$ $\widetilde{X}^v(x) = (x, X^v(x))$

$s_v(x) = (x, v)$

Notice then we have that φ transfers the trivial section to the section induced by LIVFs

$$\begin{aligned} \varphi(s_v(x,v)) &= \varphi(x,v) \\ &= (x, d_e l_x(v)) \\ &= (x, X^v(x)) \\ &= \widetilde{X}^v(x) \end{aligned}$$

The following diagram commutes

$$\boxed{\begin{tikzcd} G \times T_eG \arrow[rr, "\varphi"] \arrow[dr, shift left] & & TG \arrow[dl, shift left] \\ & G \arrow[ul, shift left, "s_v"] \arrow[ur, shift left, "\widetilde{X}^v"'] & \end{tikzcd}}$$

- So a LIVF is induced by a trivial section of the bundle (interpreting book.)

9/13/06.3

- Assume X, Y are vector fields on M then $X \circ Y$ is not a vector field on M, but is a 2$^{\text{nd}}$ order differential operator. $(X, Y \neq 0)$

- X can be regarded as a derivation of $C^\infty(M) \subseteq \underline{C^\infty_{loc}(P)}$

some comment here really working with germ.

$$X(f + cg) = X(f) + c\,X(g)$$

$$X(fg)(x) = f(x)\,X_x(g) + g(x)\,X_x(f)$$

as we know tangent vectors can be identified as ~~a~~ curves, derivations or tensors, but here we take the derivation so anyway,

$$X(f)(x) = X_x(f)$$

$$X \in \mathrm{End}(C^\infty(M)) \leftarrow \text{associative algebra } (+, \cdot, \circ)$$

$$X \in \mathrm{Der}(C^\infty(M)) \leftarrow (+, \cdot \text{ but not } \circ)$$

$$\begin{aligned}
(X \circ Y)(f) &= X(Y(f)) \\
&= \sum_{i=1}^{m} X^i \frac{\partial}{\partial x^i}(Y(f)) \\
&= \sum_{i=1}^{m} X^i \frac{\partial}{\partial x^i}\left(\sum_{j=1}^{m} Y^j \frac{\partial f}{\partial x^j}\right) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{m}\left(X^i \frac{\partial Y^j}{\partial x^i}\frac{\partial f}{\partial x^j} + \underline{\underline{X^i Y^j}} \frac{\partial^2 f}{\partial x^i \partial x^j}\right)
\end{aligned}$$

2$^{\text{nd}}$ order derivatives.

However we see that if we consider $[X, Y] = X \circ Y - Y \circ X$ then the 2$^{\text{nd}}$ order terms cancel and $[X, Y] \in \mathrm{Der}(C^\infty(M))$. See pg. 147 for coordinate free argument.

Conclusions:

(1.) $End(C^\infty(M))$ is an associative algebra under $+, \cdot, \circ$.

(2.) $Der(C^\infty(M))$ is a subspace of $End(C^\infty(M))$ and $X, Y \in Der(C^\infty(M)) \Rightarrow [X, Y] \in Der(C^\infty(M))$

(3.) If $(\mathcal{a}, +, \cdot, \circ)$ is an associative algebra and if we define $[a, b] = a \circ b - b \circ a$ then $(\mathcal{a}, +, \cdot, [\,,\,])$ is a Lie Algebra

Def^n/ Lie algebra means that $\mathcal{a} \times \mathcal{a} \xrightarrow{[\,,\,]} \mathcal{a}$ is

(1.) bilinear

(2.) $[x, y] = -[y, x]$

(3.) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$

$\forall x, y, z \in \mathcal{a}$. (In general $\mathcal{a}$ need not be an associative algebra but when it is commutator bracket gives Lie structure.

(4.) $(Der(C^\infty(M)), +, \cdot, [\,,\,])$ is a Lie algebra.

Examples

(1.) $(\mathbb{R}^3, +, \cdot, \times)$

$$(X, Y) \longrightarrow [X, Y] = X \times Y$$

(2.) $(gl(n), +, \cdot, \circledast)$ matrix multiplication.

$$[A, B] = AB - BA$$

(3.) $(\Gamma(M), +, \cdot, [\,,\,])$ is a Lie algebra.

Def^n/ Let M, N be manifolds and $\varphi : M \to N$ a smooth mapping. If $X \in \Gamma(M)$ and $Y \in \Gamma(N)$ then we say X is φ-related to Y iff $\forall x \in X$

$$d_x \varphi (X_x) = Y_{\varphi(x)}$$

Lemma: Let M, N be manifolds and let $\varphi : M \to N$ is a smooth mapping. Assume that $X_1, X_2 \in \Gamma(M)$ and $Y_1, Y_2 \in \Gamma(N)$. If X_i is φ-related to Y_i for $i = 1, 2$ then $[X_1, X_2]$ is φ-related to $[Y_1, Y_2]$

Remark: A zero-dim'l manifold M $x : U \to x(U) \subseteq \mathbb{R}^0 = \{0\}$ So then U better be a singleton. So there must a chart for every point. Also $T_p M = \{0\}$

$$f : M \to N \implies df = 0$$

oh, so pretty much everything is trivial.

M, N manifolds

$\varphi : M \longrightarrow N$

$X \in \Gamma M$, $Y \in \Gamma N$

$X \overset{\varphi}{\sim} Y$ (not an equivalence relation, read left $\to$ right)

$$\boxed{d_x \varphi (X) = Y_{\varphi(x)}} \quad \text{(I)}$$

$$d\varphi (X(x)) = Y(\varphi(x))$$

$$d\varphi \circ X = Y \circ \varphi$$

Take a C^∞ fnct in nbhd of $\varphi(x)$, $g \in C^\infty(N)$

$$Y_{\varphi(x)}(g) = d_x \varphi(X_x)(g) = X_x(g \circ \varphi)$$

That is $\boxed{Y_{\varphi(x)}(g) = X_x(g \circ \varphi)}$ (II)

Yet another way to write it,

$$Y(g)(\varphi(x)) = X(g \circ \varphi)(x)$$

All he's saying over & over is $Z(h)(x) = Z_x(h)$,

$$\boxed{Y(g) \circ \varphi = X(g \circ \varphi)} \quad \text{(III)}$$

Remark: in handwritten notes (I), (II) & (III) are integrated into argument that follows on page 2.

$$\boxed{\text{Lemma: } X_1 \overset{\varphi}{\sim} Y_1, \quad X_2 \overset{\varphi}{\sim} Y_2 \quad \text{implies} \quad [X_1, X_2] \overset{\varphi}{\sim} [Y_1, Y_2]}$$

$g \in C^\infty(N)$ then consider

$$
\begin{aligned}
d_x\varphi([X_1, X_2])(g) &= [X_1, X_2](g \circ \varphi) \\
&= (X_1)_x(X_2(g\circ\varphi)) - (X_2)_x(\overbrace{X_1(g\circ\varphi)}^{\text{must be a function.}}) \\
&= (X_1)_x(Y_2(g)\circ\varphi) - (X_2)_x(Y_1(g)\circ\varphi) \qquad \text{III} \\
&= (Y_1)_{\varphi(x)}(Y_2(g)) - (Y_2)_{\varphi(x)}(Y_1(g)) \qquad \text{II} \\
&= [Y_1, Y_2]_{\varphi(x)}(g)
\end{aligned}
$$

$\therefore\ d_x\varphi([X_1, X_2]) = [Y_1, Y_2]_{\varphi(x)} \quad \therefore\ [X_1, X_2] \overset{\varphi}{\sim} [Y_1, Y_2].$

Thm/ If G is a Lie group and $X \in \Gamma_{inv}(G)$ and $Y \in \Gamma_{INV}(G)$ then $[X, Y] \in \Gamma_{inv}(G)$

Proof: Recall $Z \in \Gamma_{inv}(G) \iff d_x l_a(Z_x) = Z_{l_a(x)}$

So Z is left-invariant iff $Z \overset{l_a}{\sim} Z$ $\forall a \in G$.

If X, Y are LIVF then

$$X \overset{l_a}{\sim} X$$
$$Y \overset{l_a}{\sim} Y$$

By the lemma $[X, Y] \overset{l_a}{\sim} [X, Y]$

therefore $[X, Y] \in \Gamma_{INV}(G)$.

Cor/ $\Gamma_{INV}(G)$ is a sub-Lie algebra of $\Gamma(G)$

Proof: a sub-Lie algebra is a subset which is also a Lie algebra, $\Gamma_{inv}(G)$ satisfies that since $X, Y \in \Gamma_{INV}(G) \implies$

$$d_x l_a(X + cY) = d_x l_a(X) + c\, d_x l_a(Y)$$

linearity of $d_x l_a$ insures vector space prop.

$$d_x l_a(X + cY) = X_{ax} + cY_{ax}$$
$$= (X + cY)_{ax}$$

$$\therefore X + cY \in \Gamma_{INV}(G).$$

by Thm we know $[\Gamma_{INV}(G), \Gamma_{INV}(G)] \subset \Gamma_{INV}(G)$.

Def^{n}/ The Lie algebra of a Lie group G is the $\Gamma_{INV}(G)$ as a Lie algebra under the vector field brackets just described.

- Recall $\exists$ a bijection from T_eG onto $\Gamma_{INV}(G)$ defined by $V \overset{\Phi}{\longmapsto} X^V$ where

$$X^V(x) = d_e l_x(V)$$

and $\Phi : T_eG \longrightarrow \Gamma_{INV}(G)$ is a vector space isomorphism. We define the bracket operation on T_eG by $[V,W]$ is the unique element of T_eG such that

$$X^{[V,W]} \overset{def^n}{\equiv} [X^V, X^W]$$

for all $V, W \in T_eG$.

$$[V,W] \equiv \Phi^{-1}\left([\Phi(V), \Phi(W)]\right)$$

Remark: Jacobi identity is not very hard but there is some work here.

Therefore: $(T_eG, +, \cdot, [\,,\,]) \cong (\Gamma_{INV}(G), +, \cdot, [\,,\,])$ Both are called the Lie algebra of G. There is yet another characterization of $\underline{\underline{g}}$ in terms of the exponential.

Lie Algebra of G

More on Matrix Groups

9/15/06.5

Consider a global chart

$$X_{ij}(A) = A_{ij} \quad \text{or} \quad x^i_j(A) = A^i_j$$

this is a global chart on $gl(n)$ well ok (X_{ij}) is the chart on $gl(n)$. Now $Gl(n) \subset_{open} gl(n)$ and $gl(n)$ is vector space so we identify $T_A Gl(n) = gl(n)$.

$$B \in gl(n) \longleftrightarrow \sum_{i,j} B_{ij}\left(\frac{\partial}{\partial X_{ij}}\bigg|_A\right) \in T_A Gl(n)$$

If $f: Gl(n) \longrightarrow Gl(n)$ is a smooth mapping then $d_A f : T_A Gl(n) \to T_{f(A)} Gl(n)$ is defined on $V = \sum_{i,j} B_{ij} \frac{\partial}{\partial X_{ij}}\Big|_A$ by transforming V to a vector $W = \sum_{i,j} D_{ij} \frac{\partial}{\partial X_{ij}}\Big|_{f(A)}$ where the components of V are transformed to components of W via the Frechet Derivative of the local coord. rep of f.

generally

$$\begin{tikzcd}
v \in T_A M \arrow[r, "d_A f"] \arrow[d] & T_{f(A)} N \ni w \arrow[d] \\
\mathbb{R}^m \arrow[r, "D(y \circ f \circ x^{-1})"] & \mathbb{R}^m
\end{tikzcd}$$

$(v^1, v^2, \ldots, v^m) \qquad\qquad (w^1, \ldots, w^m)$

$$D_A(X \circ f \circ X^{-1})(B) = D$$

$$(D_A f)(B) = D$$

Now specialize f to $f = l_C$. This is the restriction of a linear mapping & we know Frechet derivative of linear map is itself thus

$$(D_A l_C)(B) = l_C(B) = CB$$

Therefore

$$d_I l_C : \underset{\underset{gl(n)}{\wr\wr}}{T_I Gl(n)} \longrightarrow \underset{\underset{gl(n)}{\wr\wr}}{T_C Gl(n)}$$

$$(d_I l_C)(B) = CB$$

or thinking of B as a derivation,

$$d_I l_C \left(\sum B_{ij} \frac{\partial}{\partial x^{ij}}\Big|_I \right) = \sum_{i,j} (CB)_{ij} \frac{\partial}{\partial x^{ij}}\Big|_C$$

$$= \sum_{i,j,k} C_{ik} B_{kj} \frac{\partial}{\partial x^{ij}}\Big|_C$$

$$= \sum_{i,j,k} X_{ik}(C) B_{kj} \frac{\partial}{\partial x^{ij}}\Big|_C$$

Then LHS $= \overline{X}^B(C)$ $\therefore$ $\boxed{\overline{X}^B = \sum_{i,j,k} X_{ik} B_{kj} \frac{\partial}{\partial x^{ij}}}$

$B \in gl(n)$ view as tangent vector $B = \sum B_{ij} \frac{\partial}{\partial x^{ij}}\Big|_I$

$$\boxed{X^B = \sum_{i,j,k} x_{ik} B_{kj} \frac{\partial}{\partial x^{ij}}}$$

• this is a LIVF so if it holds at I it holds everywhere.

Wish to show that $[X^{B_1}, X^{B_2}] = X^{[B_1, B_2]}$

$$X_I^{B_1}(X^{B_2}(f)) = X_I^{B_1}\Big(\sum_{i,j,k} B_{2kj} x_{ik} \frac{\partial f}{\partial x^{ij}}\Big)$$

$$= \sum_{a,b} B_{1ab}\Big(\frac{\partial}{\partial x^{ab}}\Big|_I\Big)\Big(\sum_{i,j,k} B_{2kj}\, x_{ik} \frac{\partial f}{\partial x^{ij}}\Big)$$

$$= \Big(\sum_{a,b}\sum_{i,j,k} B_{1ab} B_{2kj} \frac{\partial x_{ik}}{\partial x^{ab}} \frac{\partial f}{\partial x^{ij}}$$

$$+ \sum_{a,b}\sum_{i,j,k} B_{1ab} B_{2kj}\, x_{ik} \frac{\partial^2 f}{\partial x^{ab}\partial x^{ij}}\Big)\Big|_{\text{at } I}$$

$$= \sum_{i,j,k} B_{1ik} B_{2kj} \frac{\partial}{\partial x^{ij}}(f)\Big|_I + \sum_{a,b}\sum_{i,j} B_{1ab} B_{2ij} \frac{\partial^2 f}{\partial x^{ab}\partial x^{ij}}\Big|_I$$

Likewise,

$$X_I^{B_2}(X^{B_1}(f)) = \sum_{i,j,k} B_{2ik} B_{1kj} \frac{\partial}{\partial x^{ij}}(f)\Big|_I + \sum_{a,b}\sum_{i,j} B_{2ab} B_{1ij} \frac{\partial^2 f}{\partial x^{ab}\partial x^{ij}}\Big|_I$$

Therefore noting the 2nd order terms cancel,

$$[X^{B_1}, X^{B_2}]_I(f) = \sum_{i,j}\big[(B_1 B_2) - (B_2 B_1)\big]_{ij} \frac{\partial f}{\partial x^{ij}}(I)$$

$$= \cancel{x} \sum_{i,j} x_{ij}([B_1, B_2]) \frac{\partial}{\partial x^{ij}}\Big|_I f$$

$$= X_I^{[B_1, B_2]}(f)$$

This true $\forall f \in C^\infty_{loc}(I) \Rightarrow$ under left translation $\Rightarrow$ equal as vect. fields.

$\therefore$ the bracket on $T_I\, gl(n)$ is the comm. bracket.

(comments on defn on pg. 156) by ☆ 9/18/06.2

> Defn/ Let G be a Lie group and $H \subseteq G$. We say H is a Lie subgroup of G iff it is a Lie group such that
>
> (1.) H is a subgroup of G
>
> (2.) H is a submanifold of G

Remark: often (2.) is weakened to allow immersed submanifolds. However it is quite strong since H closed (topologically) in $G \Rightarrow H$ submanifold. The irrational wind around torus $\approx x \to (e^{ix}, e^{i\alpha x})$

> Defn/ If H is a Lie group and G is a Lie group and $i : H \longrightarrow G$ is an injective immersion and group homomorphism then we say H is an injectively immersed subgroup of G (well $i : H \longrightarrow G$ really)

Remark: $i : H \longrightarrow G$ an injective immersion implies i and $d_p i$ are injective $\forall p \in H$.

Remark: one would like a correspondance between Lie groups & Lie algebras and subalgebras & subgroups. Clearly plain old submanifolds miss certain subalgebras as in the irrationally wound torus

Assume $i: H \longrightarrow G$ is an injectively immersed subgroup of G. Let $v \in T_e H$ and then $d_e i(v) \in T_e G$. Let $X_H^v(x) = d_e l_x(v) \; \forall x \in H$. be the LIVF determined by v as usual.

Then $X_G^{d_x i(v)}$ is the LIVF on G determined by $d_x i(v)$

$$\boxed{\text{Proposition: If } v \in T_e H \text{ then } X_H^v \overset{i}{\sim} X_G^{d_e i(v)}}$$

Proof: $x \in H$

$$
\begin{aligned}
X_G^{d_e i(v)}(i(x)) &= d_e l_{i(x)}^G (d_e i(v)) \\
&= d_e (l_{i(x)}^G \circ i)(v) \\
&= d_e (i \circ l_x^H)(v) \\
&= d_x i(d_e l_x^H (v)) \\
&= d_x i(X_H^v(x))
\end{aligned}
$$

$(d_x \varphi (X_x) = Y_{\varphi(x)}$ was φ relatedness $)$

- the notes on pg-157 were done for submanifolds not immersed submanifolds thus the confusion
- seems to be correct on pg-158.

9/18/06.4

Th$^{\underline{m}}$/ Let G be a Lie group and let $i : H \to G$ be an injectively immersed Lie subgroup of G then $\mathfrak{h}$ may be identified as a subalgebra of $\mathfrak{g}$ where $\mathfrak{h}$ and $\mathfrak{g}$ are the Lie algebras of H and G respectively.

Proof: Sketch,

$$X_H^{v} \overset{i}{\sim} X_G^{d_e i(v)}$$

$$X_H^{w} \overset{i}{\sim} X_G^{d_e i(w)}$$

$$\left.\begin{array}{l} [X_H^{v}, X_H^{w}] \overset{i}{\sim} [X_G^{d_e i(v)}, X_G^{d_e i(w)}] \\ X_H^{[v,w]_H} \overset{i}{\sim} X_G^{[d_e i(v), d_e i(w)]_G} \end{array}\right\} \text{def}^{\underline{n}} \text{ of Lie algebra on tangent space to identity.}$$

$$\left.\begin{array}{l} d_x i\left(X^{[v,w]_H}(x)\right) = X_G^{[d_e i(v), d_e i(w)]_G}(i(x)) \\ d_e i([v,w]_H) = [d_e i(v), d_e i(w)] \end{array}\right\} x = e$$

$$(T_e H, [\,,\,]_H) \xrightarrow{d_e i} (T_e G, [\,,\,]_G)$$

$d_e i$ is an injective isomorphism, it imbeds $T_e H$ into $d_e i(T_e H) \subset T_e G$.