Next Friday Homework

1.) Do the part of 8 on page 11 which shows that

$$Gl(n, \mathbb{C}) \cong U(n) \times \mathbb{R}^{n^2}$$

as manifolds

2.) In problem 12 page 11 show

$$\left.\begin{aligned} O(2n+1) &= SO(2n+1) \times \mathbb{Z}_2 \\ O(2n) &= SO(2n) \times \mathbb{Z}_2 \end{aligned}\right\} \text{as Lie groups}$$

actually $SO(2n+1) \times \{I, I$

semi-direct prod.

M^i_j ← row ← column

take $M = [u_1 | u_2 | \cdots | u_n]$

do Gramm-Schmidt to ortogonalize then to $[V_1 | V_2 | \cdots | V_n]$ find Scalars $[C]$

$$[u_1 | u_2 | \cdots | u_n] = [V_1 | \cdots | V_n] \underbrace{\Big[\quad C \quad\Big]}_{}$$

triangular matrix with real diagonals.

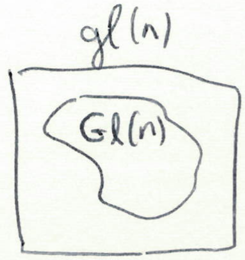
- Later will use that $U(n)$, $\mathbb{R}^{n^2}$ connected implies $Gl(n, \mathbb{C})$ connected as it is the cartesian product of connected sets. That's why we're looking at these

Theorem

There exists an open set Θ about 0 in $gl(n)$ such that $\exp(\Theta)$ is open in $Gl(n)$ and

$$\exp : \Theta \longrightarrow \exp(\Theta)$$

is a diffeomorphism, thus



$gl(n)$ Banach Algebra

$Gl(n)$ open-dense subset of $gl(n)$.

$$T_I Gl(n) = gl(n)$$

$$\Rightarrow \exp : T_I Gl(n) \longrightarrow Gl(n)$$

maps lines onto curves in $Gl(n)$ which are one-parameter groups. In Riemmannian Geometry map down to Geodesics, they will "rule" the nbhd of point. Likewise one-parameter groups rule the nbhd of identity. Abstractly still works when $G \neq$ matrix group.

$$\boxed{\left(\exp(\Theta), (\exp|_{\Theta})^{-1}\right) \text{ is a chart of } Gl(n) \text{ onto } \Theta \subseteq gl(n)}$$

- we be able to restrict the exponential to get charts on subgroups of $Gl(n)$.

Proof:

$$f(A) = \exp(A)$$

$$f(0+H) - f(0) = H \Rightarrow D_0 f(H) = H$$

$$\|f(0+H) - f(0) - H\| = \|\exp(H) - I - H\|$$

$$= \left\| \sum_{k=2}^{\infty} \frac{1}{k!} H^k \right\|$$

$$\leq \sum_{k=2}^{\infty} \frac{1}{k!} \|H\|^{k-2} \|H\|^2$$

$$= \sum_{k=0}^{\infty} \frac{1}{(k+2)!} \|H\|^k \|H\|^2$$

$$\leq \sum_{k=0}^{\infty} \frac{1}{k!} \|H\|^k \|H\|^2$$

$$= e^{\|H\|} \|H\|^2$$

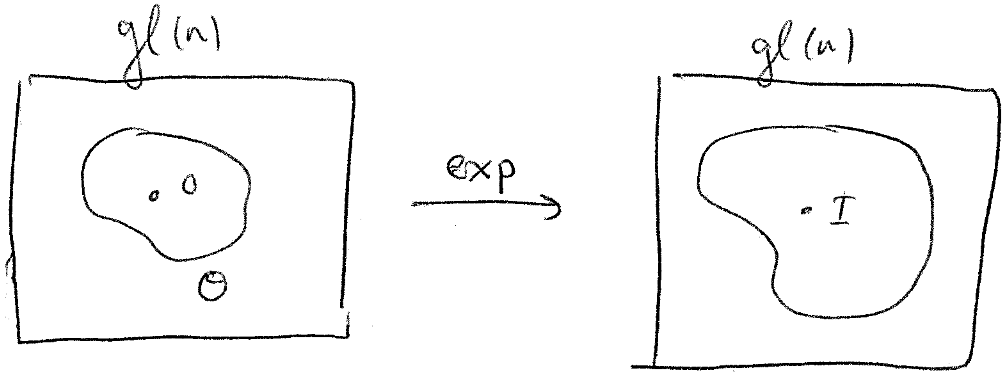
$$0 \leq \frac{\|f(0+H) - f(0) - H\|}{\|H\|} \leq e^{\|H\|} \|H\| \to 0 \text{ as } H \to 0$$

$\Rightarrow \boxed{D_0 f = Id}$ then the diff of exp at zero is invertible since $D_0 f = id_{gl(n)}$ and by inverse function Th$^{\underline{m}}$ there exists Θ open in $gl(n)$ about zero such that $\exp(\Theta)$ open in $gl(n)$ and $f|_{\Theta}$ is a diffeomorphism.

Proof: (continued)

$f(0) = I \in Gl(n)$ = open

$\Rightarrow \exists\ \hat{\Theta} \subseteq \Theta$ and $f(\hat{\Theta}) \subseteq Gl(n)$.



$f(\hat{\Theta}) \subseteq Gl(n)$

$f|_{\hat{\Theta}}$ restriction to open set open

Alternatively: $\det(\exp(A)) = \exp(\mathrm{Trace}(A))$

Chevalley Proves it in this old book Trace = Spur.

Remark: we'll prove $gl(n)$ Lie Algebra of $Gl(n)$

and so

$(\exp|_{\Theta})^{-1} : Gl(n) \longrightarrow gl(n)$

$\exp : gl(n) \longrightarrow Gl(n)$

algebra exponentiates to Group

(looking ahead.)

Corollary:

If $\mathfrak{g}_J = \{ B \in gl(n) \mid B^t J + JB = 0 \}$

then $\exists$ open set $\hat{O}$ about 0 in $gl(n)$ such that

$$\exp(\underbrace{\mathfrak{g}_J \cap \hat{O}}) = \exp(\hat{O}) \cap G_J$$

in subspace topology of $\mathfrak{g}_J \subset gl(n, \mathbb{R})$

and $\exp(s\mathfrak{g}_J \cap \hat{O}) = \exp(\hat{O}) \cap (SG_J)$

where s & S stand for special meaning

$$s\mathfrak{g}_J = \{ B \in \mathfrak{g}_J \mid \text{Trace}(B) = 0 \}$$

$$SG_J = \{ A \in G_J \mid \det(A) = I \}$$

again we require $J^2 = I$, $J^t = J$

Remark: at a point a metric can be diagonalizer to $\pm \delta_{ij}$ this J gadget includes $\pm$ cases.

$J = I \longrightarrow O(n) = G_I$

$J = I \longrightarrow SO(n) = SG_I$.

$J = \begin{pmatrix} 1 & \\ & -I \end{pmatrix} \longrightarrow G_J$ = Lorentz Group

SG_J = connected comp of identity

- Replace t with daggers $\dagger = (t)^*$ and prove it in the $\mathbb{C}$ - case.

Proof:

Let Θ be open in $gl(n, \mathbb{C})$ such that $\exp(\Theta)$ open in $Gl(n, \mathbb{C})$ and $\exp|_\Theta$ is diffeomorphism

Choose $\hat{\Theta}$ open about zero such that $\hat{\Theta} \subset \Theta$

and $B \in \hat{\Theta} \Rightarrow \bar{B}, B^T, B^\dagger, -B, JBJ^{-1} \in \Theta$

more over choose $\hat{\Theta}$ such that if $A, B \in \hat{\Theta}$

$\Rightarrow A + B \in \Theta$

all of these are continuous mappings from $gl(n) \to gl(n)$ thus we can choose a small enough subset of Θ

Claim One:

$$\exp\left(\mathfrak{g}_J^{\mathbb{C}} \cap \hat{\Theta}\right) = G_J^{\mathbb{C}} \cap \exp(\hat{\Theta})$$

$$A \in G_J^{\mathbb{C}} \cap \exp(\hat{\Theta}) \iff \begin{pmatrix} A \in \exp(B) & , B \in \hat{\Theta} \\ A^\dagger J A = J & \end{pmatrix}$$

$$\iff \begin{pmatrix} A \in \exp(B) & , B \in \hat{\Theta} \\ \exp(B)^\dagger J \exp(B) = J & \end{pmatrix}$$

$$\iff \begin{pmatrix} A \in \exp(B) & B \in \hat{\Theta} \\ \exp(B^\dagger) J = J \exp(-B) & \end{pmatrix}$$

$$\iff \begin{pmatrix} A \in \exp(B) & B \in \hat{\Theta} \\ \exp(B^\dagger) = J \exp(-B) J^{-1} & \end{pmatrix}$$

$$\iff \begin{pmatrix} A \in \exp(B) & B \in \hat{\Theta} \\ \exp(B^\dagger) = \exp(-JBJ^{-1}) & \end{pmatrix}$$

$$\iff \begin{pmatrix} A \in \exp(B) & B \in \hat{\Theta} \\ B^\dagger = -JBJ^{-1} & \end{pmatrix} \iff \begin{pmatrix} A \in \exp(B) & B \in \hat{\Theta} \\ B^\dagger J + JB = 0 & \end{pmatrix}$$

Proof continued:

$$\Longleftrightarrow \quad A \in \exp\left(\mathfrak{g}_J^{\mathbb{C}} \cap \hat{\theta}\right)$$

Claim Two:

#12 some comments

$$O(2n+1) \cong \underbrace{SO(2n+1) \times \mathbb{Z}_2}_{\text{really the cross product.}} \quad \text{where} \quad \mathbb{Z}_2 \cong \{-1, 1\} \subset \mathbb{R}^\times.$$

$$A \longrightarrow \left(\frac{1}{\det A} A \,,\, \det(A) \right)$$

as a manifold $O(2n+1)$ is the union of two connected sets $SO(2n+1) \times \{1\} \cup SO(2n+1) \times \{-1\}$, then should check the group isomorphism.

$\boxed{\text{Def}^{\underline{n}}}$/ A, B Lie groups and $\tau : B \to \mathrm{Aut}(A)$ is a group homomorphism we define a group

$$A \times_\tau B$$

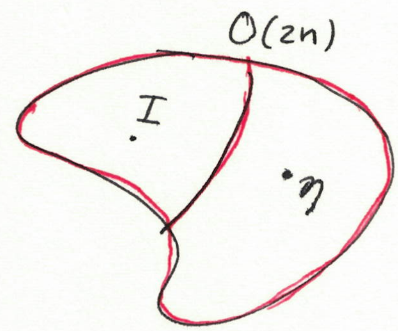
$$(a_1, b_1) \cdot (a_2, b_2) \equiv (a_1 \tau(b_1)(a_2) \,,\, b_1 b_2)$$

with identity (e_A, e_B)

Remark: $O(2n) \cong SO(2n) \times_\tau \{I, \eta\}$ is what $O(2n) \cong SO(2n) \times \mathbb{Z}_2$ really means and τ is the inner automorphism

$$\tau(z)(A) = z A z$$

$$B \longrightarrow \begin{cases} (B, I) & B \in SO(2n) \\ (B\eta, \eta) & B \in SO(2n)\eta \end{cases}$$



#3, 4, 5 pg. 10 next friday

Matrix Groups continued

$$\exp\left(\mathfrak{g}_J^{\mathbb{C}} \cap \hat{\Theta}\right) = G_J^{\mathbb{C}} \cap \exp(\hat{\Theta})$$

$0 \in \hat{\Theta} \subseteq gl(n, \mathbb{C})$ and $\exp|_{\Theta}$ diffeomorphism

$\hat{\Theta} < \Theta$ where $B \in \hat{\Theta} \Rightarrow B^*, B^\dagger, B^T, JBJ^{-1} \in \hat{\Theta}$

smaller set & $\hat{\Theta} + \hat{\Theta} \subseteq \Theta$, $|\mathrm{Tr}\, B| < \frac{\pi}{2}$

we knew we could choose such an $\hat{\Theta}$ by continuity of the operations in question. ~~To~~

To prove the exp statement we used a few identities,

$$e^{A+B} = e^A e^B \quad \text{provided} \quad AB = BA$$

in particular we see,

$$I = e^0 = e^{B-B} = e^B e^{-B} \quad \therefore \quad \left(e^B\right)^{-1} = e^{-B}$$

Also we used

$$\left(e^B\right)^\dagger = e^{B^\dagger} \quad \text{since} \quad \left(B^k\right)^\dagger = \underbrace{B^\dagger B^\dagger \cdots B^\dagger}_{k} = \left(B^\dagger\right)^k$$

And finally

$$A e^B A^{-1} = e^{ABA^{-1}} \quad \text{since} \quad \left(ABA^{-1}\right)^k = A B^k A^{-1}.$$

$$e^{ABA^{-1}} = \sum_{k=0}^{\infty} \frac{\left(ABA^{-1}\right)^k}{k!} = A\left(\sum_{k=0}^{\infty} \frac{B^k}{k!}\right)A^{-1} = A e^B A^{-1}$$

9/6/06. 3

$$\boxed{\underline{\text{Claim 2}}: \exp\left(s\,\mathfrak{g}_J^{\mathbb{C}} \cap \hat{\Theta}\right) = s\,G_J^{\mathbb{C}} \cap \exp(\hat{\Theta})}$$

infinitesimally preserve metric

· then $\exp|_{\hat{\Theta}}$ will give chart structure on $s\,\mathbb{G}_J^{\mathbb{C}}$ which will include the example $SU(n)$ and more.

$$
\begin{aligned}
A \in s\,G_J \cap \exp(\hat{\Theta}) &\iff A \in G_J \cap \exp(\hat{\Theta}) \ \& \ \det(A) = 1 \\
&\iff A \in \exp(\mathfrak{g}_J) \cap \exp(\hat{\Theta}) \ \& \ \det(A) = 1 \\
&\iff A \in \exp(\mathfrak{g}_J \cap \hat{\Theta}) \ \& \ \det(A) = 1 \\
&\iff A = \exp(B) \ \& \ B \in \mathfrak{g}_J \cap \hat{\Theta} \ \& \ \det(A) = 1 \\
&\iff A = \exp(B),\ B \in \hat{\Theta},\ B^\dagger J + JB = 0,\ \det(A) = 1 \\
&\iff A = \exp(B),\ B \in \hat{\Theta},\ \det(A) = 1,\ B^\dagger = -JBJ \\
&\iff \left(\text{By } \star \ \det(e^B) = e^{\operatorname{trace}(B)} = e^0 = 1\right)
\end{aligned}
$$

Now lets split up and in math into $\mathbb{R}/\mathbb{C}$.

a.) in the real case

$\det(A) = 1 \iff \operatorname{trace}(B) = 0$

b.) in complex case

$\det(A) = 1 \iff \operatorname{trace}(B) = 2\pi m i$ for $m \in \mathbb{Z}$.

Choice of $\hat{\Theta} \Rightarrow |\operatorname{Tr} B| < \pi/2 \Rightarrow m = 0 \Rightarrow \operatorname{trace}(B) = 0$.

Therefore

$$\boxed{A \in s\,G_J \cap \exp(\hat{\Theta}) \iff \exp(B) = A,\ B \in \hat{\Theta},\ \operatorname{Trace}(B) = 0 \ \& \ B^\dagger = -JBJ}$$

$$A \in sG_J \cap \exp(\hat{\Theta}) \iff A = \exp(B), \quad B \in \hat{\Theta}, \quad \text{Trace}(B) = 0, \quad B^\dagger = -JBJ$$

$$\iff A = \exp(B), \quad B \in \hat{\Theta}$$

$$B \in s\mathfrak{g}_J \cap \hat{\Theta}$$

$$\iff A \in \exp(s\mathfrak{g}_J \cap \hat{\Theta}).$$

Proof of $\det(\exp B) = \exp(\text{trace}(B))$. The easy way.

Proof: $f(t) = \det(e^{tB})$

$$f'(t) = \frac{d}{dh}\left(f(t+h)\right|_{h=0}$$

$$= \frac{d}{dh}\det\left(e^{(t+h)B}\right)\Big|_{h=0}$$

$$= \frac{d}{dh}\det(e^{tB})\det(e^{hB})\Big|_{h=0}$$

$e^{(t+h)B} = e^{tB}e^{hB}$

$$= \det(e^{tB})\frac{d}{dh}\det e^{hB}\Big|_{h=0}$$

$$= \det(e^{tB})\frac{d}{dh}\left[\det(I + hB + \tfrac{1}{2}h^2B)\right]\Big|_{h=0}$$

can drop since will $\to 0$ under evaluation $h=0$.

$$= \det(e^{tB})\frac{d}{dh}\det\begin{bmatrix} 1+hB_1^1 & hB_2^1 & \cdots & hB_n^1 \\ hB_1^2 & 1+hB_2^2 & \cdots & hB_n^2 \\ \vdots & & & \vdots \\ \vdots & & \cdots & 1+hB_n^n \end{bmatrix}\Bigg|_{h=0}$$

$$= \det(e^{tB})\frac{d}{dh}\left((1+hB_1^1)(1+hB_2^2)\cdots(1+hB_n^n)\right)\Big|_{h=0}$$

$$= \det(e^{tB})\sum_{i=1}^{n}(1+hB_1^1)\cdots\frac{d}{dh}(1+hB_i^i)\cdots(1+hB_n^n)\Big|_{h=0}$$

$$= \det(e^{tB})\,\text{Tr}(B) = \text{Tr}(B)\,f(t).$$

$$f'(t) = Tr(B) f(t) \qquad f(0) = 1$$

$$g(t) = e^{Tr(B)t}$$

$$g'(t) = Tr(B) e^{(tr B)t} = Tr(B)\, g(t) \quad g(0) = 1.$$

So $f(t) = g(t)$ $\forall t$ by uniqueness Th$^{\underline{m}}$ thus

$$\det(e^{tB}) = e^{(\text{Trace } B)t}$$

$$\boxed{\det(e^{B}) = e^{\text{Trace}(B)}}$$

page 10: 3,4,5

for #3 consider continous map $x \mapsto gxg^{-1}$ usually σ_g for FULP.
take nbhd of e $\mathcal{U}$ then shrink to connected open set V
so that $V^2 \subset \mathcal{U}$ and $V = V^{-1}$ can do by $W, W^{-1} \subseteq U$
and $W \cap W^{-1}$ will be its own inverse then

can show $\bigcup_{n=1}^{\infty} V^n \equiv H$ is an open & closed subgroup

$$V^2 = \bigcup_{a \in V} aV \quad \text{union of open is open}$$

$$\cdots \Rightarrow \bigcup_{n=1}^{\infty} V^n \quad \text{is open}$$

look at cosets of H these will be open and complement H.
may assume connected set is only open & closed set

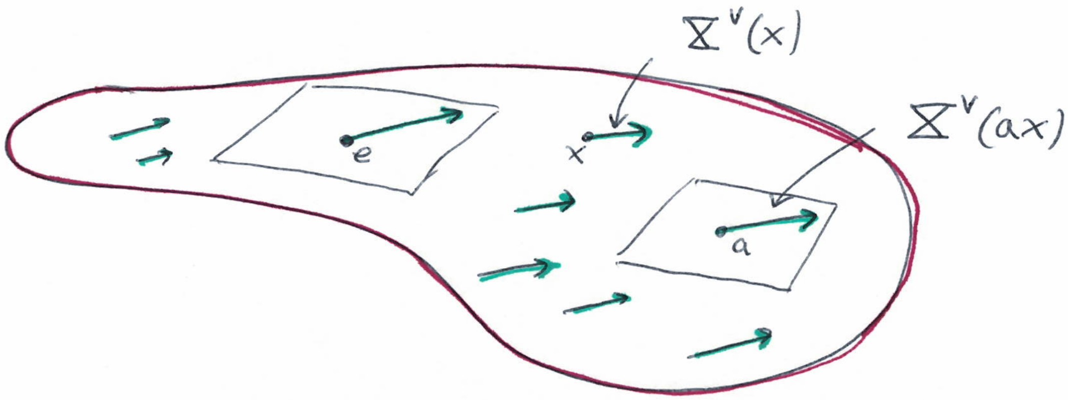
HINT: FOR #5 monday

Th^{m}/ Let G be a Lie Group and $v \in T_eG$. Define a map $X^v : G \to T_eG$ by

$$X^v(x) = d_e l_x(v)$$

Then X^v is a smooth vector field and

$$d_x l_a(X^v(x)) = X^v(ax)$$



$$l_a(y) = ay$$
$$l_a(e) = a$$

$$d_e l_a : T_eG \to T_aG$$

- this says S^2 – sphere cannot be a Lie Group since these LIVF are non vanishing throughout the Lie group.

- to show $X = \sum a^i_s(x) \frac{\partial}{\partial x^i}$ is smooth we show that $a^i(x)$ are smooth, e.g. have continuous partial derivatives of all orders. alternatively show $X(f)$ is smooth for all smooth f, this is equivalent in finite dim'l case & forms def^n in ∞-dim'l case. To see equivalence consider $f = x^i$.

Def^n/ $g \in C^\infty_{loc}(p) \iff \exists$ open set U about p such that $g \in C^\infty(U)$.

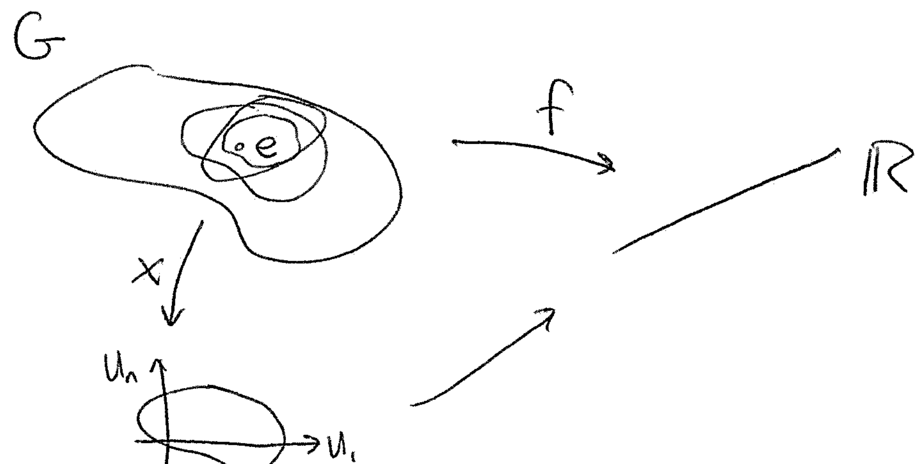
(What is difference between this and $g \in \text{Germ}(p)$?)
anyway we just work locally since we don't want to bump-up things. It is sufficient to show $f \in C^\infty_{loc}(p) \Rightarrow \mathfrak{X}f \in C^\infty_{loc}(p)$ to show $\mathfrak{X}$ smooth. (Manifolds refresher finished)

—//—

Proof: First prove $\mathfrak{X}^v$ is smooth. To do this we show if $f \in C^\infty_{loc}(e)$ then

$$\mathfrak{X}^v(f) \in C^\infty_{loc}(e)$$

Let $(\tilde{U}, x)$ be a chart at e. Let $U \subseteq \tilde{U}$ be open such that $e \in U$, $U^2 \subseteq \tilde{U} \cap \text{dom}(f)$



Is $\mathfrak{X}^v(f) \circ x^{-1}$ smooth?

Proof continued

$$[\mathfrak{X}^V(f)\circ x^{-1}](r) = \mathfrak{X}^V(f)(x^{-1}(r))$$
$$= \mathfrak{X}^V_{x^{-1}(r)}(f)$$
$$= d_e l_{x^{-1}(r)}(V)(f)$$
$$= df_{x^{-1}(r)}(d_e l_{x^{-1}(r)}(V)) \quad : \text{ since } w(f) = df(w).$$
$$= d_e(f\circ l_{x^{-1}(r)})(V) \quad : \text{ chain-rule}$$
$$= V(f\circ l_{x^{-1}(r)})$$
$$= V^i \frac{\partial}{\partial x^i}\Big|_e (f\circ l_{x^{-1}(r)})$$
$$= V^i \frac{\partial}{\partial u^i}(f\circ l_{x^{-1}(r)}\circ x^{-1})\Big|_{u=0}$$

$f\circ l_{x^{-1}(r)}\circ x^{-1} : x(U) \to f(l_{x^{-1}(r)}(U))$. Moreover

$$(r, u) \longrightarrow f(m(x^{-1}(r), x^{-1}(u))) \leftarrow \text{smooth fnct of } r \,\&\, u$$

is smooth on $x(U)\times x(U)$ now $m(U\times U) = U^2 \subset \tilde{U}$ inside the chart domain and dom(f). The map

$$r \longmapsto \frac{\partial}{\partial u^i}\left[f(m(x^{-1}(r), x^{-1}(u)))\right]\Big|_{u=0}$$

is smooth thus

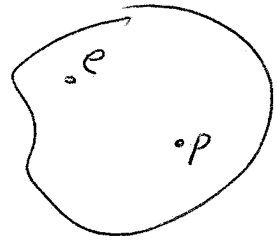
~~$\frac{\partial}{\partial r^i}$ might be better notation of instead look at (u, r) and leave $\frac{\partial}{\partial u^i}$~~

$$\mathfrak{X}^V(f)\circ x^{-1} \in C^\infty_{loc}(0)$$

oops $\therefore \mathfrak{X}^V(f) \in C^\infty_{loc}(e)$.

Proof continued

Let $f \in C^\infty_{loc}(P)$



We show that

$$\mathbb{X}^V(f) \in C^\infty_{loc}(P)$$

by showing that $\mathbb{X}^V(f) = \mathbb{X}^V((f \circ l_p) \circ l_p^{-1})$

actually $$\underbrace{\mathbb{X}^V(f)}_{\text{function}} = \underbrace{\mathbb{X}^V(\underbrace{f \circ l_p}_{C^\infty_{loc}(e)})}_{\text{just proved smooth}} \circ \underbrace{l_p^{-1}}_{\text{smooth}}$$

$$\begin{aligned}
\mathbb{X}^V(f \circ l_p)(q) &= \mathbb{X}^V_q(f \circ l_p) \\
&= d_e l_q(V)(f \circ l_p) \\
&= d_q(f \circ l_p)(d_e l_q(V)) \\
&= d_e(f \circ l_p \circ l_q)(V) \\
&= d_e(f \circ l_{pq})(V) \\
&= d_{pq} f(d_e l_{pq}(V)) \\
&= d_{pq} f(\mathbb{X}^V(pq)) \\
&= \mathbb{X}^V(f)(pq) \\
&= \mathbb{X}^V(f)(l_p(q)) \\
&= (\mathbb{X}^V(f) \circ l_p)(q)
\end{aligned}$$

$\therefore \mathbb{X}^V(f \circ l_p) = \mathbb{X}^V(f) \circ l_p$

$\Rightarrow \mathbb{X}^V(f) = \mathbb{X}^V(f \circ l_p) \circ l_p^{-1}$