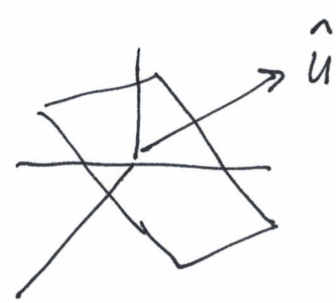
today handed out new installment of hand written notes pg. 157 → 194

Eijk very powerful, "Levi Civita" maybe

Hints: on the $so(3) \cong \{\mathbb{R}^3, \times\}$

$$\varphi\begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} = (x, y, z)$$

$A\hat{u} = \hat{u}$ $\qquad \|\hat{u}\| = 1$



$\hat{u}^{\perp}$ is a plane in $\mathbb{R}^3$

$\langle Ax, Ay \rangle = \langle x, y \rangle$

restrict A to the plane then A is a rotation around this axis in $\hat{u}$ direction

(1.) $X \in so(3)$

$Xu = 0 \qquad u = \varphi(X)$

(2.) $e^{tX} u = u \qquad u = \varphi(X)$

$f(t) = e^{tX} u - u$

also $\|X\| = 2\|\varphi(x)\|$

(3.) $\varphi(X)$ is the <u>unique</u> vector upto a sign such that

$e^{tX}\varphi(X) = \varphi(X)$ $\qquad$ (#2 proves this, the question in #3 is uniqueness.

(4.) $e^{AXA^{-1}} A\varphi(X) = A\varphi(X)$

$\Rightarrow \varphi(AXA^{-1}) = \pm A\varphi(X)$

Thm/ Let G be a Lie group and $i : H \longrightarrow G$ an immersed Lie subgroup of G. Let $\mathfrak{h}$ be the Lie algebra of LIVF on H and $\mathfrak{g}$ be the Lie alg. of LIVF on G. Then $\mathfrak{h}$ may be identified as a sub Lie algebra of $\mathfrak{g}$.

Proof:

Let $\Phi : T_e H \longrightarrow \mathfrak{h}$ & $\Psi : T_e G \longrightarrow \mathfrak{g}$ be the vector space isomorphisms

$$\Phi(v) = X_H^v \qquad \Psi(w) = X_G^w$$

we have defined $[\,,\,]_H$ on $T_e H$ and $[\,,\,]_G$ on $T_e G$ such that the following hold due to the defn $[X^v, X^w] \equiv X^{[v,w]}$.

$$\Phi([v_1, v_2]_H) = [X_H^{v_1}, X_H^{v_2}]$$

$$\Psi([w_1, w_2]_G) = [X_G^{w_1}, X_G^{w_2}]$$

Now $\Phi^{-1} : \mathfrak{h} \longrightarrow T_e H$ is a Lie alg. isomorphism. by the fact Φ an isomorphism & defn of brackets. Also from last time or pg. 157

$$d_e i : T_e H \xrightarrow{\text{(into)}} T_e G \quad \text{ok}$$

is a Lie algebra ~~isomorphism~~ onto $d_e i(T_e H) \subset T_e G$

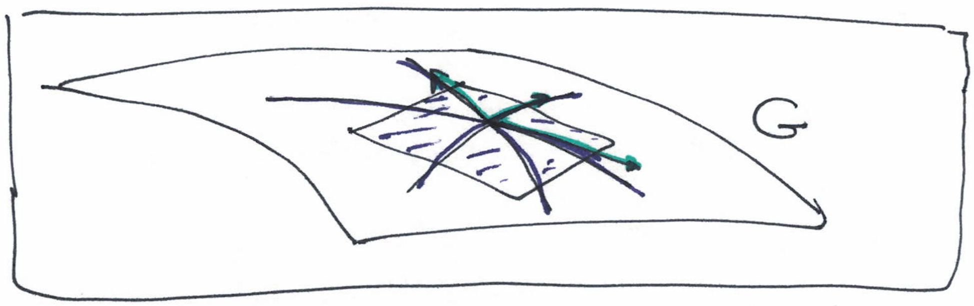
So then since each of the following is a Lie algebra homomorphism (?)

$$\Psi \circ d_e i \circ \Phi^{-1} : \mathfrak{h} \xrightarrow{\Phi^{-1}} T_e H \xrightarrow{d_e i} T_e G \xrightarrow{\Psi} \mathfrak{g}$$

is an Lie alg. isomorphism from $\mathfrak{h}$ onto a sub Lie algebra of $\mathfrak{g}$

Let $i : G \longrightarrow Gl(n)$ be an <u>immersed</u> Lie subgroup of $Gl(n)$. Then $T_e G \cong \mathfrak{g}$ is identified as a subspace of $T_I Gl(n) \cong gl(n)$. So suppose we have such a G then how do we find $T_I G$

$$T_I G = \{ A \in gl(n) \mid A = \gamma'(0) \text{ for some curve } \gamma : (-a,a) \rightarrow G, \ \gamma(0) = I \}$$



The Lie algebra $\mathfrak{g}$ of G is $\{ X^B \mid B \in T_I G \}$

$$[X^{B_1}, X^{B_2}] = X^{[B_1, B_2]} \quad \leftarrow \text{true } \forall B_1, B_2 \in gl(n) \ \& \ gl(n) = T_I Gl(n).$$

$$B_1, B_2 \in T_I G \Rightarrow [B_1, B_2] \in T_I G$$

Cor. on pg. 160

<u>Example</u>: Find Lie algebra of $O(n)$. Take a curve $\gamma : (-a,a) \longrightarrow O(n)$ $\quad \gamma(t)^T \gamma(t) = I \ \ \forall t.$

Of course $O(n)$ is a submanifold hence its an immersed subman.

Let $\gamma(\lambda) = A_\lambda$ for him $A_\lambda^T A_\lambda = I$

$$\gamma'(t)^T \gamma(t) + \gamma(t)^T \gamma'(t) = 0$$

<u>t=0</u> $\quad \gamma'(0)^T \gamma(0) + \gamma(0)^T \gamma'(0) = 0$

Now call $\gamma'(0) \equiv B \in T_I O(n) \equiv o(n)$ we find $B^T = -B$

- G a Lie group

$$\mathfrak{g} = \ell(G) = \Gamma_{inv}(G) = T_e(G)$$

$\varphi : G \longrightarrow H$: a smooth homomorphism of Lie groups.

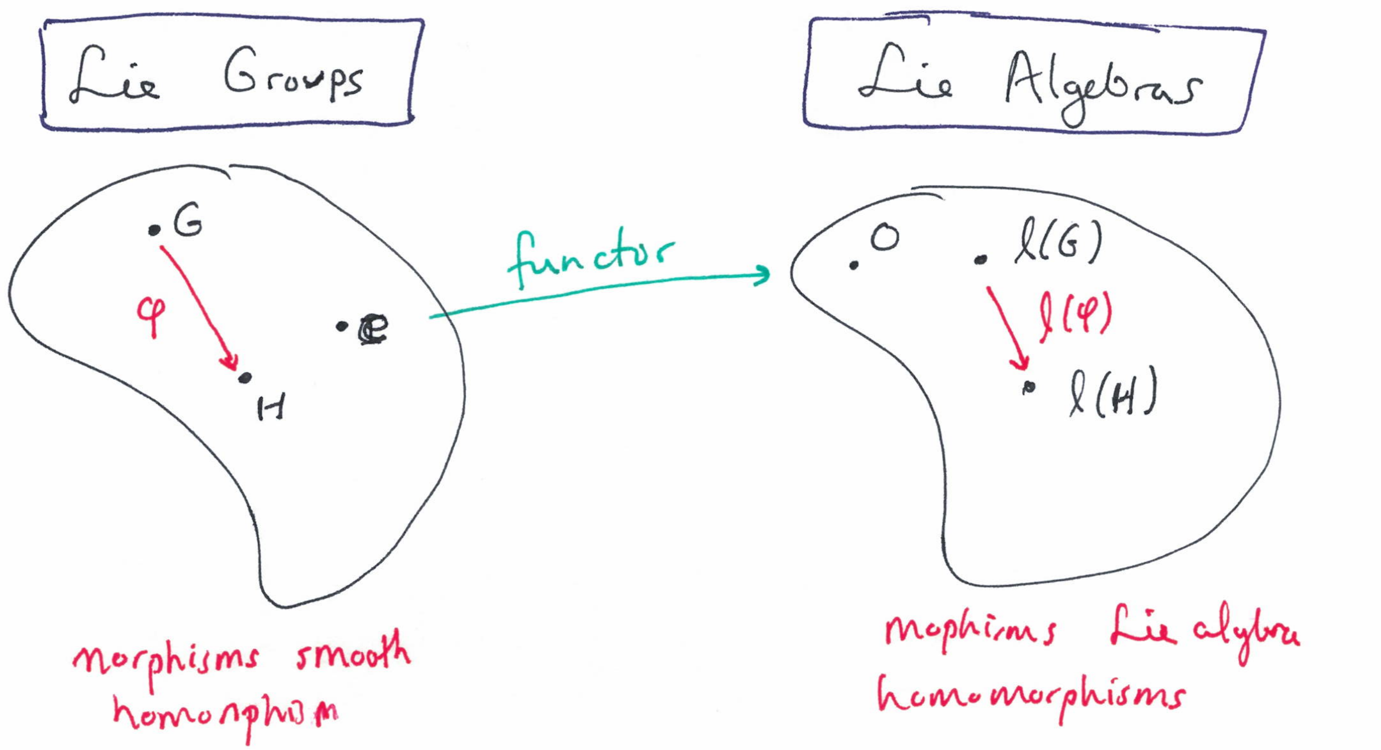
$d_e\varphi : T_eG \longrightarrow T_eH$: claim, this is a Lie algebra homomorphism.

[Read the notes or duplicate the proof for $i : H \longrightarrow G$ except we've no injectiveness here.]

$$d_e\varphi([V_1, V_2]) = [d_e\varphi(V_1), d_e\varphi(V_2)]$$

$$\tilde{\varphi} : \mathfrak{g} \longrightarrow \mathfrak{h}$$

$$\tilde{\varphi}(X_G^V) = X_H^{d_e\varphi(V)}$$ (same thing in other notation)



Hwk #1, 2 pg. 22

$\ell(G) = T_e G$ (for this $\ell(\varphi) = d_e \varphi$)

$\ell(G) = \Gamma_{inv}(G)$ (for this $\ell(\varphi) = \tilde{\varphi} : \Gamma_{inv}(G) \to \Gamma_{inv}(H)$ where $\tilde{\varphi}(X_G^v) = X_H^{d_e\varphi(v)}$)

Thm/ If G & H are Lie groups and $\varphi : G \to H$ is a Lie group homomorphism then $\ell(\varphi) : \ell(G) \to \ell(H)$ is a Lie algebra homomorphism. Also $\varphi : G \to H$ and $\psi : H \to K$ are Lie group homomorphisms then $\ell(\psi \circ \varphi) = \ell(\psi) \circ \ell(\varphi)$.

__Proof:__ $(\varphi \circ l_a^G)(x) = \varphi(ax) = \varphi(a)\varphi(x)$

$$d_x\varphi(X_G^v(x)) = d_x\varphi(d_e l_x^G(v))$$
$$= d_e(\varphi \circ l_x^G)(v)$$
$$= d_e(l_{\varphi(x)}^H \circ \varphi)(v)$$
$$= d_e l_{\varphi(x)}^H(d_e\varphi(v))$$
$$= X_H^{d_e\varphi(v)}(\varphi(x)) \quad \therefore \quad X_G^v \overset{\varphi}{\sim} X_H^{d_e\varphi(v)}$$

$(\varphi \circ l_x)(a) = \varphi(xa) = \varphi(x)\varphi(a) = (l_{\varphi(x)} \circ \varphi)(a)$

Also note,

$$[X_G^v, X_G^w] \overset{\varphi}{\sim} [X_H^{d_e\varphi(v)}, X_H^{d_e\varphi(w)}]$$

$$X_G^{[v,w]_G} \overset{\varphi}{\sim} X_H^{[d_e\varphi(v), d_e\varphi(w)]_H}$$

$$d_x\varphi(X_G^{[v,w]}(x)) = X^{[d_e\varphi(v), d_e\varphi(w)]}(\varphi(x))$$

$$d_e\varphi([v,w]) = [d_e\varphi(v), d_e\varphi(w)]$$

$\ell(\varphi) = d_e\varphi$ under this identification.

what if we use the other identification, Proof continued:

$$\tilde{\varphi}([X_G^v, X_G^w]) = \tilde{\varphi}(X_G^{[v,w]})$$
$$= X_H^{d_e\varphi([v,w])}$$
$$= X_H^{[d_e\varphi(v),\, d_e\varphi(w)]}$$
$$= [X_H^{d_e\varphi(v)}, X_H^{d_e\varphi(w)}] \quad \text{def}^n$$
$$= [\tilde{\varphi}(X_G^v), \tilde{\varphi}(X_G^w)].$$

$\varphi : G \longrightarrow H$

$\psi : H \longrightarrow K$

$\psi \circ \varphi : G \longrightarrow H$

$d_e(\psi\circ\varphi) = d_e\psi \circ d_e\varphi$

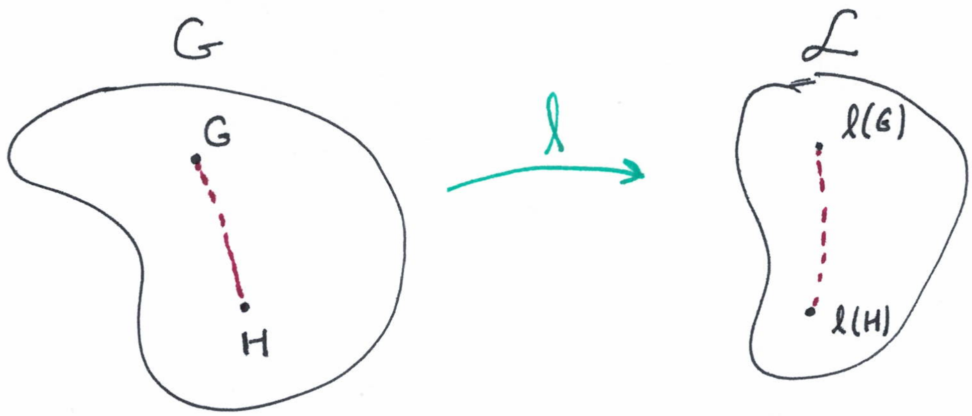
$\ell(\psi\circ\varphi) = \ell(\psi)\circ\ell(\varphi)$

Notice,

$$(\widetilde{\psi\circ\varphi})(X_G^v) = X_K^{d_e(\psi\circ\varphi)(v)}$$
$$= X_K^{d_e\psi(d_e\varphi(v))}$$
$$= \tilde{\psi}(X_H^{d_e\varphi(v)})$$
$$= (\tilde{\psi}\circ\tilde{\varphi})(X_G^v) \quad \therefore \ \widetilde{\psi\circ\varphi} = \tilde{\psi}\circ\tilde{\varphi}$$

Also we can see since $d_e\varphi = id_{T_eG}$ so $\ell(\varphi) = id_{\ell(G)}$.
Moreover $\ell(\varphi\circ\varphi^{-1}) = \ell(\varphi)\circ\ell(\varphi^{-1}) = \ell(Id) = Id_{\ell(G)}$.

9/22/06.3



Sketch of how to prove onto:

$$\mathfrak{g} \xrightarrow[\text{Th}^{\text{m}}]{\text{Ado's}} \mathfrak{gl}(n) = \ell(Gl(n))$$

$$\mathfrak{h} \subseteq \ell(G)$$

$H \subseteq G \qquad \ell(H) = \mathfrak{h}$ need Frobenius Th$^{\text{m}}$ to get H from G...

If G & H are simply connected.

$$\ell(G) = \ell(H) \Rightarrow G = H$$

otherwise,

$$\ell(G) = \ell(H) \Rightarrow \exists \text{ discrete } D \subset G \text{ such that } G/D \cong H$$

need covering theory to aptly describe.

(<u>Skip</u> 166 – 183 for now)

183–185 taken from older notes L instead of ℓ

One-Parameter "subgroups" of $Gl(n)$

$$\gamma(t) = e^{tA}$$

$$\gamma : (\mathbb{R}, +) \longrightarrow Gl(n)$$

$$\gamma(t+s) = e^{(t+s)A} = e^{tA}e^{sA} \qquad \text{since } [tA, sA] = 0.$$

A one parameter group is a mapping, not a subgroup.

$$X^B(C) = \sum C_{ik} B_{kj} \frac{\partial}{\partial x_{ij}}\bigg|_C$$

$$gl(n) \cong T_C\, Gl(n)$$

$$X^B(C) \longleftrightarrow CB.$$

$$\gamma_B(t) = e^{Bt}, \quad X^B(\gamma_B(t)) = \gamma_B(t)B = e^{tB}B = \gamma_B'(t).$$

this says the one-parameter group is a sol^{n} of the DEq^{n} $\gamma_B'(t) = X^B(\gamma_B(t))$ where $\gamma_B(0) = I$ moreover $e^B = \gamma_B(1)$ this is our motivation for the abstract group exponential which will be defined to give the same properties for arbitrary Lie group.

K. Ball, N. Philips, G. Dolphin, N. Pearson, S. Holzhaus — 80
Alex - 53

1• Need to use 1-parameter groups. For the 1st one take a 1-param. group, look at its Kernel, you can prove the Ker. has to be discrete. $\lambda : \mathbb{R} \to G$

$$\operatorname{Ker} \lambda \subseteq \mathbb{R}$$

either $\operatorname{Ker}(\lambda) = \{0\}$ or $\exists$ a least positive # then can prove by contradiction... can prove everything is a multiple of smallest # $\Rightarrow \simeq \mathbb{Z}$. Then he'll allow us to use $G/\text{discrete} \to H$ smooth... $\Rightarrow$ circle.

2• May need the fact if $f : \mathbb{R}^n \to \mathbb{R}^m$ is a homeomorphism then $n = m$. Likewise $f : U \xrightarrow[\text{homeo.}]{\text{onto}} f(U)$ then $n = m$.

$U \subset \mathbb{R}^n$, $f(U) \subset \mathbb{R}^m$

$f : G \to H$ a bijective homomorphism which is smooth.

want to show f^{-1} smooth,

$d_e f : T_e G \to T_e H$ (injective?)

$d_e f(v) = 0 \Rightarrow v = 0$ ~~($v \neq 0$)~~

$\gamma'(0) = v$

look at $\mu_s(t) = \gamma(st)$: another 1-param. grp.

$\to \nu_s = f \circ \mu_s$: the image of μ

$\nu'(\bar{s}) = 0 \quad \forall \bar{s}$

$\Rightarrow \gamma(st) = \text{constant} = \gamma(0) = e$

set $t = 1 \Rightarrow \gamma(s) = e \Rightarrow v = 0$

$\Rightarrow d_e f$ injective at e.

$\Rightarrow df$ injective everywhere just "translate" it.

Assume $\dim(G) = \dim(H)$ so this argument works. otherwise its not obvious why $d_a f_a(U)$ is open.

$$\begin{array}{ccc} TG & \xrightarrow{df} & TH \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow[f]{} & H \end{array}$$

exp is a local diffeomorphism.

But we've not yet discussed the exp at that point in the text.

//

Th^m (10) If G is a Lie group and $V \in T_eG$ then $\exists !$ smooth mapping $\varphi : \mathbb{R} \to G$ such that

$$\frac{d\varphi}{dt}(t) = X(\varphi(t)) \quad \forall t \in \mathbb{R}$$

$$\varphi(0) = e$$

Where X is the LIVF on G such that $X(e) = V$ and also φ is a 1-parameter group

$$\varphi(t+s) = \varphi(t)\varphi(s)$$

Proof: First note if $x = (x^1, x^2, \dots, x^n)$ is a chart centered at zero ($x(e) = 0$) then the equation

$$\frac{d\varphi}{dt}(t) = X(\varphi(t))$$

Can be written in terms of the chart x,

$$X(p) = \sum_{i=1}^{n} \bar{a}^i(p) \left.\frac{\partial}{\partial x^i}\right|_p \qquad \forall p \in \operatorname{dom}(x).$$

Define $a^i(z) = \bar{a}^i(x^{-1}(z))$ so $a^i : \operatorname{Im} x \to \mathbb{R}$ also define $\varphi^i = x^i \circ \varphi : I \subseteq \mathbb{R} \to \mathbb{R}$ so we may extract ordinary DEq^n's from the manifold DEq^n, Proof continued: 9/25/06.3

Translate $\varphi'(t) = \mathbb{X}(\varphi(t))$ into the following ODE

$$\varphi'(t) = \mathbb{X}(\varphi(t)) \overset{\sim}{\iff} \sum_i \frac{d\varphi^i}{dt}\frac{\partial}{\partial x^i}\Big|_{\varphi(t)} = \sum \bar{a}^i(\varphi(t))\frac{\partial}{\partial x^i}\Big|_{\varphi(t)}$$

$$\overset{\sim}{\iff} \sum_i \frac{d\varphi^i}{dt}\frac{\partial}{\partial x^i}\Big|_{\varphi(t)} = \sum_i a^i(\varphi^1(t),\dots,\varphi^n(t))\frac{\partial}{\partial x^i}\Big|_{\varphi(t)}$$

$$\overset{\sim}{\iff} \boxed{\frac{d\varphi^i}{dt} = a^i(\varphi^1(t), \varphi^2(t), \dots, \varphi^n(t))}$$

locally ($\sim$) an ODE which is the meaning of $\varphi'(t) = \mathbb{X}(\varphi(t))$.

To solve $\varphi'(t) = \mathbb{X}(\varphi(t))$ with $\varphi(0) = e$ is equivalent to solving

$$\frac{d\varphi^i}{dt} = a^i(\varphi^1(t), \varphi^2(t), \dots, \varphi^n(t))$$

$$\varphi^i(0) = x^i(e) = 0.$$

now if the fncts. a^i are nice enuf. we can always uniquely solve the system of ODEs near zero. That is $\exists\, a > 0$ and a smooth map $\varphi : (-a, a) \to G$ such that $\varphi'(t) = \mathbb{X}(\varphi(t))$ $\forall t \in (-a, a)$. Now we can't say how big a is from ODE theory. We claim we can extend $a \to +\infty$ which is nice. We prove a lemma towards that goal.

Lemma: If φ_1 & φ_2 are defined on open intervals I_1, I_2 and (finite or infinite).

$$\varphi_1'(t) = X(\varphi_1(t)) \quad \forall t \in I_1$$

$$\varphi_2'(t) = X(\varphi_2(t)) \quad \forall t \in I_2$$

and if $\exists\ t_0 \in I_1 \cap I_2$ such that $\varphi_1(t_0) = \varphi_2(t_0)$ then $\varphi_1(t) = \varphi_2(t)\ \forall t \in I_1 \cap I_2$.

Proof: Let $S = \{t \in I_1 \cap I_2 \mid \varphi_1(t) = \varphi_2(t)\}$ where $t_0 \in S$. We show S is open. Let $t_* \in S$, choose a chart X at $p = \varphi_1(t_*) = \varphi_2(t_*)$ such that $X(p) = 0$. Let

$$X(q) = \sum \bar{a}^i(q) \left.\frac{\partial}{\partial x^i}\right|_q$$

Let $a^i = \bar{a}^i \circ X^{-1}$. We have for $j = 1, 2$

$$\frac{d\varphi_j^i}{dt} = a^i(\varphi_j^1(t), \varphi_j^2(t), \dots, \varphi_j^n(t))$$

$$\varphi_1^i(t_*) = \varphi_2^i(t_*) = 0 \qquad \forall i = 1, 2, \dots, n = \dim(G)$$

now appeal to uniqueness thm, $\exists$ an interval J about t_* such that $\varphi_1(t) = \varphi_2(t)\ \forall t \in J \subset S$

$\therefore t_*$ is an interior point of S

$\therefore$ every pt. of S is an interior point $\therefore S$ open

But S is also clearly closed & S is a subset of a connected set $\therefore S = I_1 \cap I_2$.

Vector Fields on S^2

$M = S^2$

$$\mathbb{X}(x,y,z) = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

$$\varphi : (-a,a) \longrightarrow S^2$$

satisfies $\varphi'(t) = \mathbb{X}(\varphi(t))$ iff

$$\varphi'(t) = -y(\varphi(t))\frac{\partial}{\partial x} + x(\varphi(t))\frac{\partial}{\partial y}$$

We denoted $x(\varphi(t))$ by $\varphi^1(t)$ and $y(\varphi(t)) = \varphi^2(t)$ las time. Hence or $\varphi'(t) = \frac{d\varphi^1}{dt}\frac{\partial}{\partial x} + \frac{d\varphi^2}{dt}\frac{\partial}{\partial y}$ ~~thus~~

$$\frac{d\varphi_1}{dt} = -\varphi_2(t)$$

$$\frac{d\varphi_2}{dt} = \varphi_1(t)$$

Also we'd write

$$\frac{dx}{dt} = -y = a^1(x,y)$$

$$\frac{dy}{dt} = x = a^2(x,y)$$

Exitence Theory applies to

$$\frac{d}{dx}\big(x(t), y(t)\big) = \big(a^1(x(t), y(t)), a^2(x(t), y(t)\big)$$

From last time

Assume $\mathbb{X}$ is a LIVF on a Lie group G. If T is the set of all $b \in \mathbb{R}$ such that $b > 0$ and

☆ $\exists \varphi : (-b, b) \longrightarrow G$ s.t. $\varphi'(t) = \mathbb{X}(\varphi(t))$ & $\varphi(0) = e$

we proved $T \neq \emptyset$, i.e. $\exists\, b > 0$ and φ satisfying ☆.

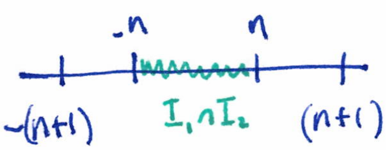
Lemma: we also proved that if φ_1, φ_2 defined on I_1, I_2 satisfy $\varphi_1'(t) = \mathbb{X}(\varphi_1(t))$ and $\varphi_2'(t) = \mathbb{X}(\varphi_2(t))$ and $\varphi_1(t_0) = \varphi_2(t_0)$ for $t_0 \in I_1 \cap I_2$ then $\varphi_1(t) = \varphi_2(t)$ $\forall t \in I_1 \cap I_2$

Case 1:

T is not bounded above. In this case we have $\forall n \in \mathbb{N}$ $\exists \varphi_n : (-n, n) \to G$ ~~such that~~ satisfying ☆ for $b = n$.

By the lemma $I_1 = (-n-1, n)$ $I_2 = (-n, n+1)$

$$\varphi_{n+1}\big|_{(-n,n)} = \varphi_n \qquad \varphi_n(0) = e,\ \varphi_{n+1}(0) = e$$



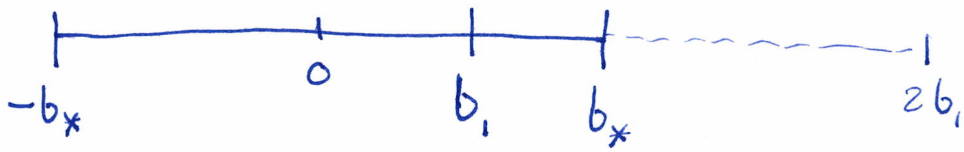
Define $\varphi : \mathbb{R} \to G$ by

$\forall t \in \mathbb{R}$, choose $n \in \mathbb{N}$ s.t. $n > t$, define $\varphi(t) = \varphi_n(t)$, φ satisfy ☆ $\varphi : \mathbb{R} \to G$ $\varphi(0) = e$ and $\varphi'(t) = \mathbb{X}(\varphi(t))$ which is what we wanted.

CASE 2:

Assume T is bounded above. Let $b_* = \text{lub}\, T = \sup T$ we seek contradiction

choose $b_1 < b_*$ s.t. $2b_1 > b_*$

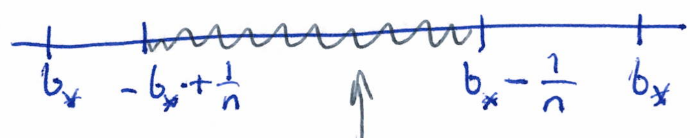


We show φ can be extended as a sol^n of (★) to $(-2b_1, 2b_1)$ but this will $\rightarrow\leftarrow$ b_* being the least upper bound since $2b_1 \in T$ and $2b_1 > b_*$ yet $2b_1 < b_*$.

Define $\tilde{\varphi} : (-b_*, 2b_1) \longrightarrow G$ by

$$\tilde{\varphi}(t) = \begin{cases} \varphi(t) & -b_* < t \le b_1 \\ \varphi(b_1)\varphi(t-b_1) & b_1 \le t < 2b_1 \end{cases}$$

[btw. We know $b_* \in T$ since we could look at



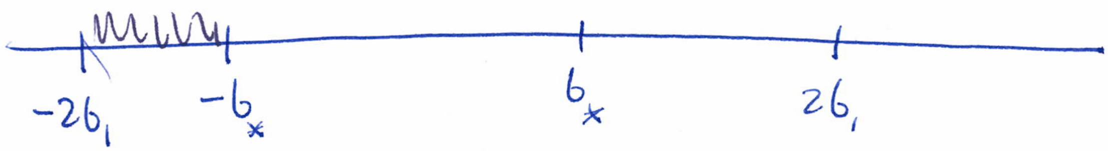
$b_* - \frac{1}{n} < b_*$ φ_n sol^n to Deq^n here

likewise for φ_{n+1} then apply Lemma to see equal so we can define $\varphi : (-b_*, b_*) \longrightarrow G$ by taking $n \longrightarrow \infty$ to get to $(-b_* \ldots)$.]

CASE 2: Continued towards →← 9/27/06.4

$$\tilde{\varphi}(t) = \begin{cases} \varphi(t) & -b_* < t \le b_1 \\ \varphi(b_1)\varphi(t-b_1) & b_1 \le t < 2b_1 \end{cases}$$

can extend to $-2b_1$ similarly he's left this out.



Note that on the interval $(0, 2b_1)$ the function $t \overset{\psi}{\longmapsto} \varphi(b_1)\varphi(t-b_1)$ is smooth and ψ satisfies $\psi'(t) = X(\psi(t))$ with $\psi(b_1) = \varphi(b_1)$. Consider that

$$-b_* < t - b_1 < b_*$$

$$\underbrace{b_1 - b_*}_{\text{negative}} < t < b_1 + b_*$$

$$2b_1 = b_1 + b_1 < b_1 + b_*$$

$$(0, 2b_1) \subseteq (b_1 - b_*, b_1 + b_*)$$

$\therefore$ ψ is defined on $(0, 2b_1)$ at least.

also we can prove ψ solves the DEq^{n}

$$\begin{aligned} \psi'(t) &= \frac{d}{dt}\left(\varphi(b_1)\varphi(t-b_1)\right) \\ &= \frac{d}{dt}\left[l_{\varphi(b_1)}(\varphi(t-b_1))\right] \\ &= d_{\varphi(t-b_1)} l_{\varphi(b_1)}\left(\frac{d}{dt}(\varphi(t-b_1))\right) \\ &= d_{\varphi(t-b_1)} l_{\varphi(b_1)}\left(X(\varphi(t-b_1))\right) \quad \text{since } X \in \Gamma_{inv.} \\ &= X(\varphi(b_1)\varphi(t-b_1)) = X(\psi(t)). \end{aligned}$$

Note that

$$\psi(b_1) = \varphi(b_1)\varphi(0) = \varphi(b_1)$$

and also (reading the little box in margin) note that φ & ψ are both defined on $(0, b_*)$ and $\psi(b_1) = \varphi(b_1)$ and the both satisfy the DEq^{2}. By the Lemma they're equal on $(0, b_*)$. it follows $\tilde{\varphi}$ is well defined, ~~therefore~~ smooth from $(-b_*, 2b_1)$ to G s.t. it satisfies $\tilde{\varphi}(0) = e$ & $\tilde{\varphi}'(t) = X(\tilde{\varphi}(t))$

a similar argument extends $\tilde{\varphi}$ to a symmetric interval about zero. namely $(-2b_1, 2b_1)$ therefore $2b_1 \in T$ yet

$2b_1 > b_*$ $\rightarrow\leftarrow$ therefore CASE 1 is the correct one and thus the sol^{n} exists for $\mathbb{R}$.

__Remark__: Now that we have φ its a one-parameter group. Fix $s \in \mathbb{R}$

$$\gamma_1 : t \longrightarrow \varphi(t+s)$$

$$\gamma_2 : t \longrightarrow \varphi(t)\varphi(s)$$

satisfy $\gamma_1(0) = \gamma_2(0) = \varphi(s)$. And they satisfy the same DEq^{n}

Remark on One-Parameter Groups Continued

$$\frac{d\gamma_1}{dt} = \frac{d}{dt}\Big(\varphi(t+s)\Big)$$

$$= \varphi'(t+s)$$

$$= X(\varphi(t+s))$$

$$= X(\gamma_1(t)).$$

$$\frac{d\gamma_2}{dt} = \frac{d}{dt}\Big[\varphi(s)\,\varphi(t)\Big]$$

$$= \frac{d}{dt}\Big[l_{\varphi(s)}\,\varphi(t)\Big]$$

$$= d_{\varphi(t)}\, l_{\varphi(s)}\,(\varphi'(t))$$

$$= d_{\varphi(t)}\, l_{\varphi(s)}\,(X(\varphi(t)))$$

$$= X(\varphi(s)\varphi(t))$$

$$= X(\gamma_2(t)).$$

Hence $\gamma_1 = \gamma_2$.

Def^{n}/ Let G be a Lie group. Let $\mathbb{X} : G \times T_eG \longrightarrow TG$ be defined by $\mathbb{X}(p, v) = \mathbb{X}^v(p) = d_e l_p(v)$. Denote the sol^{n} of $\mathbb{X}$ by $\varphi : \mathbb{R} \times T_e G \longrightarrow G$, ie

$$\frac{d\varphi}{dt} = \frac{\partial \varphi}{\partial t}(t, v) = \mathbb{X}(\varphi(t,v), v)$$

We define $\exp : T_e G \to G$ by $\exp(v) = \varphi(1, v)$

Let G be a Lie group

(1) If φ is a C^1 one-parameter group in G then φ satisfies the eq^{n}

$$\varphi'(t) = \mathbb{X}^v(\varphi(t))$$

where $v = \varphi'(0) \in T_e G$

(2) If φ is a 1-parameter group in G then $\varphi(t) = \exp(tv)$ $\forall t \in \mathbb{R}$ where $v = \varphi'(0)$.

(3.) If $v \in T_e G$ then

$$\exp((t+s)v) = \exp(tv)\exp(sv)$$
$$\exp(-v) = \exp(v)^{-1}$$

(4.) If $a \in G$ then the integral curve of $x \longmapsto \mathbb{X}^v(x)$ which passes through a is

$$t \longmapsto l_a(\exp(tv))$$

(5.) If $T_0(T_e G)$ is identified with $T_e G$ then $\exp : T_e G \longrightarrow G$ is locally invertible near $0 \in T_e G$

Remark: $\mathbb{X} : G \times T_e G \longrightarrow TG$ is smooth as a fnct. of two-variables $\Rightarrow \varphi$ is smooth. Moreover we know $\varphi(1, v)$ is defined since we proved it worked for $\mathbb{R}$ previously.