Proof of (1) thru (5)

(1). Consider $\varphi : \mathbb{R} \longrightarrow G$ a C^1, 1-parameter group.
fix any s and take derivative w.r.t. t of $\varphi(t+s)$

$$\frac{d}{dt}\left(\varphi(t+s)\right) = \frac{d}{dt}\left(\varphi(s)\varphi(t)\right)$$
$$= \frac{d}{dt}\left(l_{\varphi(s)}\left(\varphi(t)\right)\right)$$
$$= d_{\varphi(t)} l_{\varphi(s)}\left(\varphi'(t)\right)$$

$$= \varphi'(t+s)$$

Now plug in $t=0$,

$$\varphi'(s) = d_e\, l_{\varphi(s)}\left(\varphi'(0)\right) = X^{\varphi'(0)}\left(\varphi(s)\right)$$

Thus letting $\varphi'(0) = V$ we see $\varphi(0) = e$ and $\varphi'(s) = X^V(\varphi(s))$.

(2.) Let φ be a 1-parameter group in G we show
$\varphi(s) = \exp(sV)$. Def^{n} of $\exp(sV)$

$$\exp(sV) = \Theta(1, sV) = \Theta(1, sV)$$

where $\frac{d\Theta}{dt}(t) = X(\Theta(t, sV), sV) = \frac{\partial}{\partial t}\left[\Theta(t, sV)\right]$ or
in less clumsy notation $\frac{d\Theta}{dt} = X^{sV}(\Theta(t))$
and $\exp(sV) = \Theta(1)$ & $\Theta(0) = e$ (suppressing sV)

(2.) again fixing s diff. w.r.t. t

$$\frac{d}{dt}[\varphi(st)] = \varphi'(st)s$$

$$= s\,X^V(\varphi(st)) \qquad \text{where } V = \varphi'(0).$$

using 1 with ~~st~~ substituted.

$$= s\, d_{\varphi(st)} l\,(V)$$

$$= d_{\varphi(st)} l\,(sV)$$

$$= X^{sV}(\varphi(st))$$

Also $\varphi(s(0)) = e$ so this solves $\frac{d\theta}{dt} = X^{sV}(\theta(t))$
hence by uniquess thm of diff. eq's.

$$\theta(t) = \varphi(st)$$

$$\exp(sV) = \theta(1) = \varphi(s).$$

(3.) <u>If $V \in T_eG \Rightarrow \exp((t+s)V) = \exp(tV)\exp(sV)$</u> (Claim)

The soln φ of the eqn

$$\varphi'(t) = X^V(\varphi(t)), \quad \varphi(0) = e$$

is a one-parameter group then by Thm (10)

$$\varphi(t) = \exp(tV), \quad V = \varphi'(0)$$

using (2.) to conclude can write φ in terms of exp.
then we know such soln's form one-parameter group

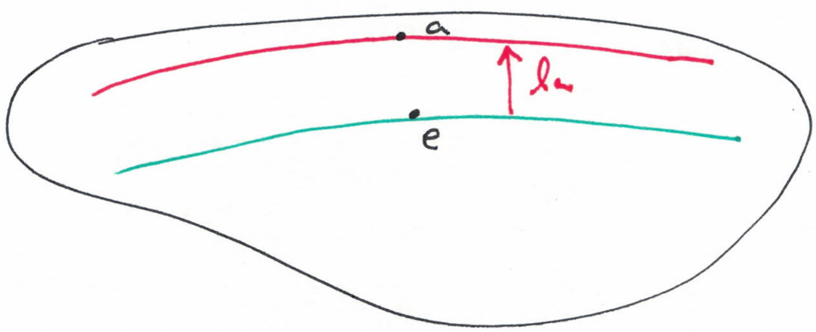
$$\exp((t+s)V) = \varphi(t+s)$$
$$= \varphi(t)\,\varphi(s)$$
$$= \exp(tV)\exp(sV)$$

the proof for the inverse prop. is similar.

<u>Remark</u>: we have defined this abstract exponential so that it recovers all the usual properties of matrix exponential

(4.) If $\psi(t) = l_a(\exp(tv))$ then

$$\psi'(t) = X^v(\psi(t)) \quad , \psi(0) = a$$



X^v's sol^{n}s.
its integral curves.

$$\begin{aligned}
\psi'(t) &= \frac{d}{dt}\left[l_a(\exp(tv)) \right] \\
&= d_{\exp(tv)} l_a \left(\frac{d}{dt} \exp(tv) \right) \\
&= d_{\exp(tv)} l_a \left(X^v(\exp(tv)) \right) \\
&= d_{\exp(tv)} l_a \left(d_e l_{\exp(tv)} (v) \right) \\
&= d_e (l_a \circ l_{\exp(tv)})(v) \\
&= d_e\, l_{a\exp(tv)} (v) \\
&= X^v(a\exp(tv)) \\
&= X^v(\psi(t))
\end{aligned}$$

Also $\psi(0) = a\exp(0) = a$.

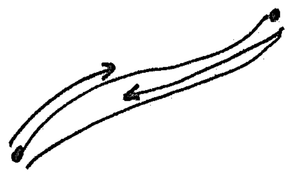
<u>we'll do (5) later.</u>

Comments on the test

9/29/06.5

gave out the review, when do we
want the test. Probably monday before break.
Also take home test # 1, 2, 4 next friday
we'll return to homework later on. You

may assume



homotopic

Local Lie Groups some History

Friday 1,2,4 on handout

Lie himself worked with special open sets so that the product was back in the set, sort of a partial group. There would be group operations when they make sense, then the idea of the global Lie group arose and the question of wether or not a local Lie group was embeddable in a global Lie group. Dr. Fulp and J. Marlin began by reading Lie's original paper in German. After some thought realized Lie's "group" was actually the Lie algebra of vector fields. Our goal in the handout is to construct a simply connected Lie group which covers an arbitrary Lie group. Also the covering group G mod the arbitrary Lie group H is G/H and is discrete. For example $O(n)/SO(n) \cong \mathbb{Z}_2$.

#

(5.) If $T_0(T_eG)$ is identified with T_eG then $d_0(\exp)$ is the identity mapping from $T_0(T_eG)$ onto T_eG. Thus $\exists$ open set $U \subseteq T_eG$ such that $0 \in T_eG$, $\exp(U)$ is open in G $\exp: U \to \exp(U)$ is a diffeomorphism.

Proof: of (5.) from pg. 190 of handwritten notes

$$\frac{d}{dt}\left[\exp(tv)\right] = d_{tv}(\exp)\left(\frac{d}{dt}tv\right) = d_{(tv)}(\exp)(v)$$

$$\varphi(t) = \exp(tv)$$

$$\varphi'(t) = X^v(\varphi(t)) \quad \text{by } (2.)$$

$$\varphi'(0) = X^v(\varphi(0)) = X^v(e) = v.$$

$$\frac{d}{dt}\left[\exp(tv)\right]\Big|_{t=0} = \varphi'(0) = v$$

$$\Rightarrow d_0(\exp)(v) = v \quad \Rightarrow \quad d_0(\exp) = Id_{T_eG}$$

Now the inverse fnct. Th^m for manifolds suggests that because $d_0 \exp$ is invertible at zero $\exists\, U$ around zero on which exp is a local diffeomorphism and many authors denote the inverse by log

$$\log \equiv (\exp|_U)^{-1}$$

of course U is not unique so this is a little sloppy.

Remark: T_xM is best linear approx to M. T_eG is best linear approx to the Group.

$$\log : \exp(U) \longrightarrow T_eG$$

gives a chart at the identity, eventually allows us to prove $C^\infty \Rightarrow$ analytic. A book for these thing Duistermaat & Kolk. He starts with a C^2 manifold and shows its analytic

we've covered 192, 193 already now we cover Th^m 12 & that's as far as it goes. The test is the review.

$\exp : T_e G \longrightarrow G$

$\exp : \Gamma_{inv}(G) \longrightarrow G$: is defined in natural way

$\exp(X) = \exp(X_e)$: just evaluate X at e to get something in $T_e G$ where we've already defined the exp.

For $X \in \Gamma_{inv}(G)$ we have a 1-parameter group

$$t \longmapsto \exp(tX) = \exp(tX_e)$$

Thm/ If G & H are Lie groups and $\varphi : G \longrightarrow H$ is a Lie group homomorphism then

$$d\varphi = \tilde{\varphi} = \ell(\varphi) : \mathfrak{g} \longrightarrow \mathfrak{h}$$

is a Lie algebra homomorphism such that the following diagram commutes,

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\ell(\varphi)"] \arrow[d, "\exp"'] & \mathfrak{h} \arrow[d, "\exp"] \\
G \arrow[r, "\varphi"'] & H
\end{tikzcd}$$

Proof:

Let $\mathfrak{X} \in \mathfrak{g} = \Gamma_{INV}(G)$. We show that

$$\varphi(\exp(\mathfrak{X})) = \exp(d\varphi(\mathfrak{X}))$$

Consider the one-parameter group in G

$$\Theta(t) = \exp(t\mathfrak{X})$$

Notice that $\varphi \circ \Theta$ is one-parameter group in H

$$\frac{d}{dt}(\varphi \circ \Theta)(t)\Big|_{t=0} = d_{\Theta(t)}\varphi\,(\Theta'(t))\Big|_{t=0}$$

$$= d_{\Theta(t)}\varphi\left(\frac{d}{dt}\left(\exp(t\mathfrak{X}_e)\right)\right)\Big|_{t=0}$$

$$= d_{\Theta(t)}\varphi\,(\mathfrak{X}_e)\Big|_{t=0}$$

$$= d_e\varphi\,(\mathfrak{X}_e)$$

$$\therefore\ (\varphi \circ \Theta)'(0) = d_e\varphi(\mathfrak{X}_e).$$

By the last Th^{m} (1—5) the one-parameter group $\varphi \circ \Theta$ is the sol^{n} of a LIVF, that is its an integral curve, it satisfies,

$$(\varphi \circ \Theta)'(t) = \mathfrak{X}^{(\varphi\circ\Theta)'(0)}((\varphi \circ \Theta)(t))$$

$$(\varphi \circ \Theta)'(0) = d_e\varphi(\mathfrak{X}_e)$$

Thus,

$$\mathfrak{X}^{(\varphi\circ\Theta)'(0)}_{(y)} = \mathfrak{X}^{d_e\varphi(\mathfrak{X}_e)}_{(y)}$$

$$= d_e l_y(d_e\varphi(\mathfrak{X}_e))$$

$$\equiv (d\varphi)(\mathfrak{X})_y \qquad \left(\begin{array}{l}\text{where } d\varphi = l(\varphi) \\ l(\varphi)(\mathfrak{X})_y = d_e l_y(d_e\varphi(\mathfrak{X}_e))\end{array}\right.$$

$\varphi \circ \Theta$ is an integral curve of the LIVF $d\varphi(X)$ on H

$$\ell(\varphi)(X) = \tilde{\varphi}(X)$$

where $\ell(\varphi)$ is the natural induced homomophism from the Lie algebra of $\mathfrak{g}$ to $\mathfrak{h}$. We found

$$X^{(\varphi\circ\Theta)'(0)}(y) = (d\varphi)(X)_y$$

We know that $(\varphi\circ\Theta)(0) = e$. By Th^{m} (11)

$$(\varphi\circ\Theta)(t) = \exp(t\, d\varphi(X))$$

$$\boxed{\varphi(\exp(tX)) = \exp(t\, d\varphi(X))}$$

When $t=1$ we get our Th^{m} but it has other uses.

Proposition: Let X be a complete vector field on a manifold M and $f: M \to N$ a diffeomorphism. Let $df(X)$ be a vector field defined by $df(X)(f(x)) \equiv d_x f(X(x))$ $\left[df(X)_y = d_{f^{-1}(y)}(X(f^{-1}(y))) \right]$ $\forall x \in M$. If $t \mapsto \varphi(t) = \varphi_t$ is the 1-parameter group determined by X then

$$t \longmapsto f \circ \varphi_t \circ f^{-1}$$

is the 1-parameter group of $df(X)$ [Recall $t \mapsto \varphi_t \Rightarrow$ in Diff(M)] $\varphi_t \in \mathrm{Diff}(M)$

Proof: Should find $f \circ \varphi_t \circ f^{-1}$ is flow of $df(X)$,

$$\frac{d}{dt}\left(f \circ \varphi_t \circ f^{-1}\right)(x) = df_{\varphi_t(f^{-1}(x))}\left(\frac{d}{dt}\left[\varphi_t(f^{-1}(x))\right]\right)$$

$$= d_{\varphi_t(f^{-1}(x))} f \left(X(\varphi_t(f^{-1}(x)))\right) \qquad \left(\begin{array}{l} \varphi_t(y) = \varphi(t,y) \\ X(\varphi_t(y)) = \dfrac{d\varphi}{dt} \end{array} \right)$$

$$= (df)(X)\left((f \circ \varphi_t \circ f^{-1})(x)\right)$$

Define $\psi(t,x) = f(\varphi(t, f^{-1}(x)))$ then observe

1.) $\frac{d}{dt}(\psi(t,y)) = df(X)(\psi(t,y))$

2.) $f \circ \varphi_0 \circ f^{-1} = f \circ id \circ f^{-1} = id$

Remark: pushforwards only for diffeomorphisms whereas pull backs are more forgiving.

$\boxed{\text{Corollary: Let } X \text{ be complete on } M \text{ and } f: M \to M \text{ a diffeomorphism. Then } df(X) = X \text{ iff } f \circ \varphi_t = \varphi_t \circ f}$

Proof: first two lines irrelevant to proof,

$$df(X) = X \iff (df)(X)(f(y)) = X(f(y))$$
$$\iff X(f(y)) = d_y f(X_y)$$
$$\iff X \overset{f}{\sim} X$$
$\iff$ pushforward of X the same as X says X is f-related to itself.

Proof: If $df(X) = X$ then they share the same flows and so the flow of X is φ_t whereas flow of $df(X)$ is $f \circ \varphi_t \circ f^{-1}$ from previous prop. thus

$$\varphi_t = f \circ \varphi_t \circ f^{-1} \quad \therefore \quad \underline{f \circ \varphi_t = \varphi_t \circ f}$$

Conversely assume $f \circ \varphi_t = \varphi_t \circ f$ $\forall t$ and recall φ_t is flow of X hence

$$X(x) = \frac{d}{dt} \varphi_t(x) \Big|_{t=0} \qquad \frac{d}{dt} [\varphi_t(x)] \Big|_{t=0} = X(\varphi_t(x)) \Big|_{t=0}$$

$$= \frac{d}{dt} (f \circ \varphi_t \circ f^{-1})(x) \Big|_{t=0}$$

$$= df(X)(x) \quad \therefore \quad X = df(X).$$

Proposition: Let X and Y be complete vector fields on a manifold M and let their one-parameter groups of diffeomorphisms be

$$t \longmapsto \varphi_t \qquad \text{AND} \qquad s \longmapsto \psi_s$$

respectively. Then $\varphi_t \circ \psi_s = \psi_s \circ \varphi_t$ $\forall t, s \in \mathbb{R}$ iff $[X, Y] = 0$.

remark: Diff(M) is ∞-dim'l Lie group where manifold structure a Frechet space, topo. given by family of semi-norms and the space of LIVF $\Gamma(M)$ on Diff(M) can be shown to be Lie algebra so we're saying in the ∞ dim'l case commutative group $\Leftrightarrow$ commutatively.

Proof: Recall that,

$$[X, Y]_x = (L_X Y)_x = -\frac{d}{dt}\Big(\overbrace{d\varphi_t(Y)_x}^{g(t)}\Big)$$

$$= \lim_{t \to 0} \frac{1}{t}\left[Y_x - d\varphi_t(Y)_x\right] \qquad \frac{1}{t}[g(0) - g(t)]$$

Now consider that

$$d\varphi_s([X, Y]_x) = \lim_{t \to 0} \frac{1}{t}\left[d\varphi_s(Y_x) - (d\varphi_s)(d\varphi_t)(Y)_x\right]$$

$$= -\frac{d}{dt}\left[d\varphi_s(d\varphi_t(Y)_x)\right]\Big|_{t=0}$$

$$= -\frac{d}{dt}\left(d(\varphi_s \circ \varphi_t)(Y)_x\right)\Big|_{t=0}$$

being careless about where "x" is, so beware of details here.

$$= -\frac{d}{dt}\left(d\varphi_{s+t}(Y)_x\right|_{t=0} \qquad \varphi_{s+t} = \varphi_{t+s}$$

$$= -\frac{d}{dt}\left(d(\varphi_t \circ \varphi_s)(Y)_x\right|_{t=0}$$

$$= -\frac{d}{dt}\left(d\varphi_t \circ d\varphi_s (Y)_x\right|_{t=0}$$

$$= -\frac{d}{dt}\left[d\varphi_t(d\varphi_s(Y)_x)\right]\Big|_{t=0} = [X, d\varphi_s(Y)]_x$$

continuing, we just saw $d\varphi_s([X,Y]_x) = [X, d\varphi_s(Y)_x]$

thus we find,

$$[X,Y]=0 \iff d\varphi_s([X,Y]) = 0$$

$$\iff [X, d\varphi_s(Y)] = 0$$

$$\iff \frac{d}{dt}\left[d\varphi_{(s+t)}(Y)_x\right]\Big|_{t=0} = 0$$

see last page.

$$\iff \frac{d}{d\tau}\left[d\varphi_\tau(Y)_x\right|_{\tau=s} = 0 \qquad \tau = t+s \quad \tau(t=0)=s.$$

$$\iff \tau \longmapsto d\varphi_\tau(Y)_x \text{ is constant fnct.}$$

$$\iff d\varphi_s(Y)_x = \text{constant}$$

$$\iff d\varphi_s(Y)_x = Y_x \qquad (\text{see } s=0 \text{ in above})$$

$$\iff d\varphi_s(Y) = Y$$

$$\iff \varphi_s \circ \psi_t = \psi_t \circ \varphi_s \quad \forall s,t.$$

using prop. on pg. 2. 10/06/06.

Proposition:
Let G be a Lie group and $X, Y \in LIVF(G)$. Let $v = X(e)$ and $w = Y(e)$, $\alpha(t) = \exp(tv)$ & $\beta(t) = \exp(tw)$. If $t \mapsto \psi_t$ and $t \mapsto \varphi_t$ are the one-parameter groups of diffeomorphisms of G determined by X and Y then

$$\varphi_t \circ \psi_s = \psi_s \circ \varphi_t \quad \forall s,t \in \mathbb{R} \iff \alpha(t)\beta(s) = \beta(s)\alpha(t) \quad \forall s,t \in \mathbb{R}.$$

Proof: Recall that the flows of X and Y are given by

$$\varphi(t,x) = l_x(\exp(tv))$$
$$\psi(t,x) = l_x(\exp(tw))$$

$$\varphi_t(x) = l_x(\exp(tv)) \qquad \varphi_t : G \longrightarrow G$$
$$\psi_t(x) = l_x(\exp(tw)) \qquad \psi_s : G \longrightarrow G$$

φ_t & ψ_s are 1-parameter group in $\mathrm{Diff}(G)$ whereas α & β are 1-parameter groups in G. Consider then,

$$\begin{aligned}(\psi_s \circ l_y)(x) &= \psi_s(yx) \\ &= l_{yx}(\exp(sw)) \\ &= l_y(l_x(\exp(sw))) \\ &= l_y(\psi_s(x)) \qquad \therefore \psi_s \circ l_y = l_y \circ \psi_s\end{aligned}$$

So we can pull out left translations, likewise $\varphi_t \circ l_y = l_y \circ \varphi_t$.
Then note

$$\begin{aligned}(\varphi_t \circ \psi_s)(y) &= (\varphi_t \circ \psi_s)(l_y(e)) \\ &= (l_y \circ \varphi_t \circ \psi_s)(e) \\ &= (l_y \circ \varphi_t)(\beta(s)) \qquad \beta(s) = \exp(sw) \\ &= (l_y \circ \varphi_t)(\beta(s)e) \\ &= (l_y \circ \varphi_t)(l_{\beta(s)}(e)) \\ &= (l_y \circ l_{\beta(s)} \circ \varphi_t)(e) \\ &= (l_y \circ l_{\beta(s)})(\alpha(t)) = y\beta(s)\alpha(t)\end{aligned}$$

Proof: continued,

$$(\varphi_t \circ \psi_s)(y) = y\,\beta(s)\,\alpha(t)$$

$$(\psi_s \circ \varphi_t)(y) = y\,\alpha(t)\,\beta(s)$$

Thus

$$\varphi_t \circ \psi_s = \psi_s \circ \varphi_t \iff y\,\beta(s)\alpha(t) = y\,\alpha(t)\beta(s) \quad \forall y$$

$$\iff \beta(s)\alpha(t) = \alpha(t)\beta(s) \,.\quad //$$

—//—

Corollary: If X & Y are LIVF(G) then $[X, Y] = 0$ iff $\exp(sw)\exp(tv) = \exp(tv)\exp(sw)$ $\forall s,t \in \mathbb{R}$. where $v = X(e)$, $w = Y(e)$.

Corollary: If G is a connected Lie group then G is Abelian iff its Lie algebra is commutative.

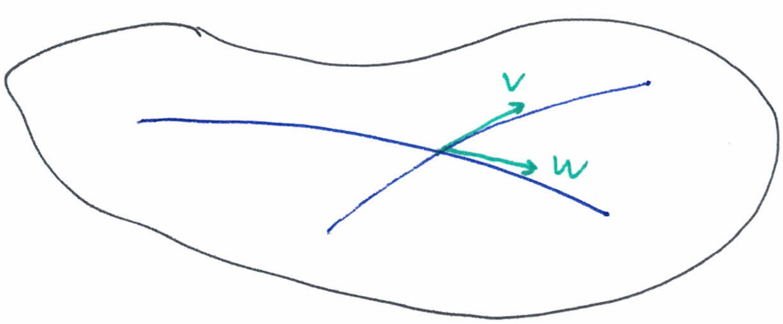
Proof: If G Abelian then $\exp(tv)\exp(sw) = \exp(sw)\exp(tv)$ $\forall s,t \in \mathbb{R}$ and $v, w \in T_eG$ therefore $[X^v, X^w] = 0$ $\forall v,w$ by previous corollary (notice $X = X^v$ and $Y = X^w$).

Conversely assume $(T_eG, [\,,\,])$ is commutative. Then we have $\exp(v)\exp(w) = \exp(w)\exp(v)$. But $\exists$ an open set U about zero in T_eG s.t. $\exp|_U : U \to \exp(U)$ is a diffeomorphism onto an open set $\exp(U) \subseteq G$. We can choose $U = -U$ also choose $V \subseteq \exp(U)$ s.t. $V = V^{-1}$, $e \in V$, V is connected. Then G is generated by elements of V, meaning

$$g \in G \Rightarrow g = V_1 V_2 \cdots V_n$$
$$\Rightarrow g = \exp(u_1)\exp(u_2)\cdots\exp(u_n) \quad \text{for } u_i \in \log(V).$$

Likewise for $h \in G$ thus $hg = gh$ since we can switch the exponentials all around.

Proposition:
Let G be a Lie group and $v, w \in T_e G$.
If $[v, w] = 0$ then
$$\exp(tv)\exp(tw) = \exp(t(v+w)) = \exp(tw)\exp(tv)$$



Proof: Let $t \mapsto \varphi_t$ and $s \mapsto \psi_s$ be one-parameter groups determined by X^v and X^w. Let
$$\mu_t = \varphi_t \circ \psi_t$$
Since $[v, w] = 0 \Rightarrow \exp(tv)\exp(sw) = \exp(sw)\exp(tv)$ ($s = t$ here.) ~~our~~ in other words φ_t and ψ_t commute thus μ_t is one parameter group of diffeomorphisms on G.
$$\begin{aligned}\mu_{t+s} &= \varphi_{t+s} \circ \psi_{t+s} \\ &= \varphi_t \circ \varphi_s \circ \psi_t \circ \psi_s \\ &= \varphi_t \circ \psi_t \circ \varphi_s \circ \psi_s \\ &= \mu_t \circ \mu_s \quad \therefore \text{ it's one parameter grp in } \mathrm{Diff}(G)\end{aligned}$$
Now which vector field determines μ?
$\exists$ a LIVF, X^u, $u \in T_e G$ such that
$$\frac{d}{dt}\left[\mu_t(g)\right] = X^u(\mu_t(g))$$
Consider then
$$\frac{d}{dt}\left(\mu_t(e)\right)\Big|_{t=0} = X^u(\mu_t(e))\Big|_{t=0} = X^u(\mu_0(e)) = X^u(e) = u.$$
which we show next that $u = v + w$.

Proof continued:

$$\mu_t(g) = l_g(\exp(tu))$$

But $\mu_t(e) = (\varphi_t \circ \psi_t)(e)$

Now for a trick, we wish to compute derivative of μ_t at $t=0$,

$$F(t_1, t_2) = (\varphi_{t_1} \circ \psi_{t_1})(e)$$

$$\begin{aligned}
\frac{d}{dt}\left[\mu_t(e)\right]\Big|_{t=0} &= \frac{d}{dt}\left[F(t,t)\right]\Big|_{t=0} \\
&= (\partial_1 F)(0,0) + (\partial_2 F)(0,0) \\
&= \frac{d}{dt}\left[\varphi_t(e)\right]\Big|_{t=0} + \frac{d}{dt}\left[\psi_t(e)\right]\Big|_{t=0} \\
&= X^v(\varphi_0(e)) + X^w(\psi_0(e)) \\
&= X^v(e) + X^w(e) \\
&= v + w \qquad \therefore \quad \underline{u = v + w}
\end{aligned}$$

in each the other is fixed at $t=0$ giving $\varphi_0 = \psi_0 = id.$

Now notice this gives us

$$\mu_t(g) = l_g(\exp(tu)) \Longrightarrow$$

$$\mu_t(e) = \varphi_t(\psi_t(e)) = \underline{\exp(tv)\exp(tw) = \exp(t(v+w))} \; //.$$