J. BROWN & J. COON (1)

P. 10

3

Let G_0 be the connected component of the identity e.

Consider the map $\sigma_g : G \to G$ where $g \in G$ is fixed

$$\sigma_g(x) = gxg^{-1}$$

25/25

we can write $\sigma_g(x) = m(g, m(x, i(g)))$

m is group multiplication and $i(g) = g^{-1}$.

Since G is a Lie Group m and i are smooth.

Therefore σ_g is smooth since it is a composition of smooth maps.

The continuous image of a connected set is connected.

Hence $\sigma_g(G_0)$ is connected.

Since $\sigma_g(e) = geg^{-1} = gg^{-1} = e$, $e \in \sigma_g(G_0)$.

Since $e \in \sigma_g(G_0)$ and $\sigma_g(G_0)$ connected then

$$\sigma_g(G_0) \subseteq G_0$$

Since G can be partitioned uniquely into connected components $\sigma_g(G_0) = G_0$, for each $g \in G$

$$\therefore \quad g G_0 g^{-1} = G_0$$

$$\therefore \quad G_0 \triangleleft G.$$

You should also ~~shown~~ show G_0 is a subgroup

$m(G_0 \times G_0) \subseteq G_0$

$i(G_0) \subseteq G_0$

why

PROBLEM 4 Show that a connected Lie group is generated by every nbhd of the unit element $e \in G$.

To begin we recall some results from set theory or ma 225, let G be a group and let $A \subseteq G$ then define the cosets of A as follows (a subgroup)

$$gA = \{x \in G \mid \exists a \in A \text{ such that } x = ga\}$$

Consider then gA and hA where $g \neq h$ in general. We argue that either $gA = hA$ or $gA \cap hA = \emptyset$. Suppose that $x \in gA$ and suppose $gA \cap hA \neq \emptyset$ then $\exists$ at least one $y \in gA \cap hA$. Thus we know,

$$y \in gA \cap hA \implies \exists a_1, a_2 \in A \text{ s.t. } ga_1 = y \ \& \ ha_2 = y$$

Considering that $x \in gA \Rightarrow \exists a_3 \in A$ s.t. $x = ga_3$ we wish to show that $x \in hA$, use the common element y to connect the cosets,

$$\begin{aligned} x &= g a_3 \\ &= y a_1^{-1} a_3 && (\text{since } ga_1 = y \Rightarrow g = ya_1^{-1}) \\ &= h a_2 a_1^{-1} a_3 && (\text{since } y = ha_2) \\ &= h a_4 && (\text{assuming that } A \text{ is a subgroup.}) \end{aligned}$$

Thus $x \in hA \therefore gA \subseteq hA$. By almost the same argument we may argue $hA \subseteq gA \therefore gA = hA$. We find that the cosets partitions the group into disjoint subsets,

$$G = \bigcup_{g \in G} gA$$

Let $\mathcal{U}$ be an open nbhd of the identity. Use the continuity of the group operations to choose an open connected set V such that $V^2 \subset \mathcal{U}$ and $V = V^{-1}$.

We can do this. ~~because we~~ can choose an open $W \subset \mathcal{U}$ then by ~~continuity of~~ ~~inverse~~ open map, it is a homeomorphism W^{-1} is open thus $W \cap W^{-1}$ is open then $V = W \cap W^{-1}$ has $V = V^{-1}$.

We show V^2 is open as follows,

$$V^2 = \bigcup_{a \in V} aV$$

So if we can show aV is open for $a \in V$ then that $\Rightarrow V^2$ open since arbitrary union of open sets is open.

Define $f_a : aV \longrightarrow V$ by $f_a(x) \equiv a^{-1}x$ then notice

$$\begin{aligned} 1.)\ f_a(aV) &= \{f_a(av) \mid v \in V\} \\ &= \{a^{-1}av \mid v \in V\} \\ &= \{v \mid v \in V\} \\ &= V. \end{aligned}$$

(into) actually onto!

Again $l_a : G \to G$ is smooth and its inverse $l_{a^{-1}}$ is also smooth. So l_a is a homeomorphism

2.) Let $V_0 \in V$ then $f(aV_0) = a^{-1}aV_0 = V_0$ (onto)

So $l_a(V)$ is open

Thus $f_a^{-1}(V) = aV$ and since f is clearly continous by smoothness of group operations it follows that aV is continous inverse image of the open set V thus aV is open thus V^2 is open.

Problem 4

Assume that V^n is open then let us show V^{n+1} is open.

$$V^{n+1} = V V^{n+1} = \bigcup_{a \in V} a V^n \quad V^{n+1} = \bigcup_{g \in V^n} (gV) \text{ is open!}$$

Note: Would not need induction although this is correct!

then using almost the same logic as before $g_a : aV^n \to V^n$ defined by $g_a(x) = a^{-1}x$ has $g_a(aV^n) = a^{-1}aV^n = V^n$

thus $g_a^{-1}(V^n) = aV^n \Rightarrow aV^n$ open since g_a continuous map.

Therefore V^{n+1} is union of open sets & hence is open.

Inductively we conclude that V^n is open for all $n \geq 1$.

Now we find $H = \bigcup_{n=1}^{\infty} V^n$ is a <u>union</u> of open sets

therefore H is open.

H is a group

1.) $a, b \in H \Rightarrow a \in V^{n_1}$ & $b \in V^{n_2}$

$\Rightarrow ab \in V^{n_1}V^{n_2} = V^{n_1+n_2} \subset H$

$\Rightarrow ab \in H.$

2.) $e \in V \Rightarrow e \in H$

3.) $a \in H \Rightarrow a \in V^n$ but $V = V^{-1} \Rightarrow V^n = (V^n)^{-1} \overset{?}{=} (V^{-1})^n$

$\Rightarrow a^{-1} \in V^n$

$\Rightarrow a^{-1} \in H.$

Thus H is a subgroup of G (we know the operations inherited from G are associative)

PROBLEM 4

We have shown H is an open connected subgroup of G. We have shown that $g_1 H \cap g_2 H = \emptyset$ unless $g_2 \in g_1 H$, the cosets of H partition G.

$$G = \bigcup_{g \in G} gH = H \cup \bigcup_{g \notin H} gH$$

$$\bigcup_{g \notin H} gH = \text{complement of of } H = \overline{H}$$

Now $f_g(x) = g^{-1}x$ where $f_g : gH \to H$ shows that $gH = f_g^{-1}(H)$ is open as f_g continous thus $\bigcup_{g \notin H} gH$ is open thus H is closed.

Then we find H is open & closed. Since G is connected H must be G since its non empty. (otherwise we'd have a seperation by H & $\overline{H}$).

$\therefore$ G is generated by $V \subset \mathcal{U}$

$\therefore$ $\boxed{G \text{ is generated by } \mathcal{U}.}$

II

6

#5 Let D be a discrete normal subgroup of a connected Lie group G.

Let $x \in G$ and $d \in D$.

Since G is connected $\exists$ curve $\alpha : I = [0,1] \to G$

$\alpha(0) = e \quad \alpha(1) = x$

Now construct the curve

$$\sigma_\alpha(t) = \alpha(t)\, d\, [\alpha(t)]^{-1}$$

note since D is normal

$$\sigma_\alpha(t) \in D \quad \forall t$$

<u>claim</u>

$$\sigma_\alpha(t) = d$$

<u>proof</u>

$D = \{d\}$ done

$D \neq \{d\}$

assume $\exists\, t_0$ s.t $\sigma_\alpha(t_0) = b \neq d \quad b \in D$

then $\tilde{\sigma}_\alpha : [0, t_\alpha] \to D$

s.t $\sigma_\alpha(t_0) = b$

$\sigma_\alpha(0) = d$

Maybe a little more efficient to say $\sigma_\alpha[0,1]$ is connected and $e \in \sigma_\alpha[0,1]$. Since D is discrete $\{d\}$ is max connected set $\therefore \sigma[0,1] = \{d\}$

OK

thus $\{b, d\} \subset \tilde{\sigma}_\alpha[0, t_\alpha] \subset D$. Note $\{b, d\}$ is disconnected but since $\tilde{\sigma}_\alpha$ is the restriction of the continuous map σ_α it is continuous thus $\tilde{\sigma}_\alpha[0, t_\alpha]$ is connected yet $\{b, d\}$ disconnected $\Rightarrow \tilde{\sigma}[0, t_\alpha]$ is disconnected $\rightarrow\leftarrow$ thus σ_α is constant mapping as claimed.

So $\sigma_\alpha(t) = d \quad \forall t$

$$\sigma_\alpha(1) = \alpha(1)\, d\, \alpha(1)^{-1} = x d x^{-1} = d$$

then $xd = dx$

we can do this $\forall x \in G$

$\Rightarrow d \in Z(G)$ the center of G.

Since d was fixed but arbitrary this implies

$$\underline{D \subseteq Z(G).}$$