JAMES COOK

COVERING MANIFOLDS II

(1.) We show $\tilde{M}$ is Hausdorff. Let $[\alpha_1], [\alpha_2] \in \tilde{M}$ with $[\alpha_1] \neq [\alpha_2]$. As before $\tilde{M}$ is the set of homotopy classes of paths that begin at x_o, so $\alpha_1(0) = \alpha_2(0) = x_o$ however $\alpha_1(1)$ and $\alpha_2(1)$ may or may not coincide.

CASE I: $\alpha_1(1) \neq \alpha_2(1)$

then since M is Hausdorff $\exists\, U_1, U_2$ such that $\alpha_1(1) \in U_1$ and $\alpha_2(1) \in U_2$ with $U_1 \cap U_2 = \emptyset$. Suppose $[\gamma] \in ([\alpha_1], U_1) \cap ([\alpha_2], U_2)$ then this implies $\exists\, \beta_1 : I \longrightarrow U_1$ & $\beta_2 : I \longrightarrow U_2$ where $[\gamma] = [\alpha_1 * \beta_1] = [\alpha_2 * \beta_2]$. Note that $\gamma(1) \in U_1$ and $\gamma(1) \in U_2$ $\therefore$ there is no such path, we find $([\alpha_1], U_1) \cap ([\alpha_2], U_2) = \emptyset$ and clearly $[\alpha_1] \in ([\alpha_1], U_1)$ & $[\alpha_2] \in ([\alpha_2], U_2)$. Hence in CASE I we find $\tilde{M}$ is Hausdorff.

CASE II: $\alpha_1(1) = \alpha_2(1) = z$, again suppose $\exists\, U_1$ & U_2 open as in CASE I. let $U = \psi^{-1}(B_\epsilon) \subset U_1 \cap U_2$ be a chart domain (B_ϵ contractible $\Rightarrow U$ contractible). Let $[\gamma] \in ([\alpha_1], U) \cap ([\alpha_2], U)$ then $\exists\, \beta_1, \beta_2 : I \longrightarrow U$ with $[\gamma] = [\alpha_1 * \beta_1] = [\alpha_2 * \beta_2]$. As customary denote reversed paths by $\tilde{\beta}_1$ and $\tilde{\beta}_2$, now both β_1 and β_2 end at $\gamma(1) \in U$. Note $\beta_1 * \tilde{\beta}_2$ and $\beta_2 * \tilde{\beta}_1$ are loops in U based at $\gamma(1)$, thus

$$[\alpha_1 * \beta_1] = [\alpha_2 * \beta_2]$$

$$\Rightarrow [\alpha_1 * \underbrace{\beta_1 * \tilde{\beta}_2}_{\text{loops shrink in contractible } U \text{ (no holes)}}] = [\alpha_2 * \beta_2 * \tilde{\beta}_2] = [\alpha_2]$$

$\therefore [\alpha_1] = [\alpha_2]$ but $[\alpha_1] \neq [\alpha_2]$ this is a contradiction.

James Coon

(2) Let $x \in M$ and let U be a contractible chart domain of M such that $x \in U$. Suppose α is a path from x_0 to x,

(a.) Let $z \in P(([\alpha], U))$ then $\exists\ [f] \in ([\alpha], U)$ such that $P([f]) = f(1) = z$. Notice $[f] \in ([\alpha], U)$ by definition $\Rightarrow \exists\ \beta : I \longrightarrow U$ with $\alpha(1) = x = \beta(0)$ and $\beta(I) \subset U$ but in particular $\beta(1) \in U$, and f is the combination of α and β, $f = \alpha * \beta$. This means $f(1) = (\alpha * \beta)(1) = \beta(1) \in U \therefore z \in U$

$$\therefore\ P(([\alpha], U)) \subseteq U.$$

Let $y \in U$ then since U contractible it follows * $\exists$ a path in U from x to y, call it $\beta : I \longrightarrow U$ $\beta(0) = x$ & $\beta(1) = y$. Note $[g] = [\alpha * \beta] \in ([\alpha], U)$ and also $P([g]) = g(1) = (\alpha * \beta)(1) = y$. Thus $y \in P(([\alpha], U))$ $\therefore$ $U \subseteq P(([\alpha], U))$

Therefore we find $P(([\alpha], U)) = U$.

*: Contractible $\Rightarrow$ shrink loops in U to a point
$\Rightarrow$ connected.
$\Rightarrow$ path connected.

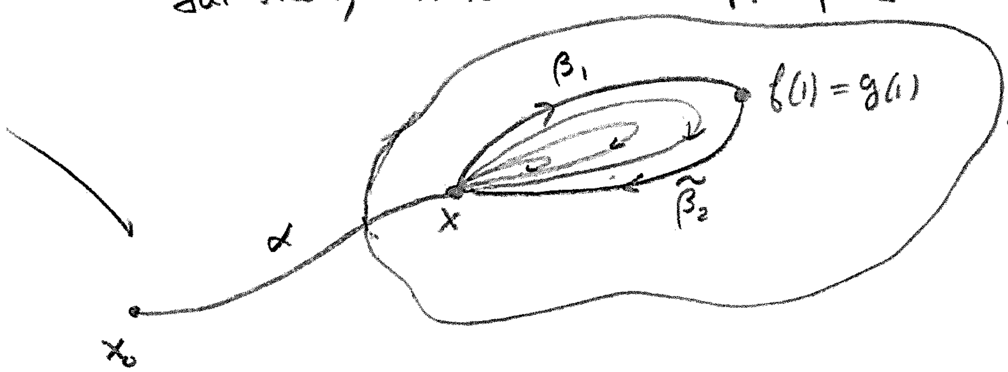
JAMES GUN

(2b) Show $P|_{U_{[\alpha]}}$ is injective. Let $[f], [g] \in ([\alpha], U)$, suppose

$$P|_{U_{[\alpha]}}([f]) = P|_{U_{[\alpha]}}([g])$$

$$\Rightarrow f(1) = g(1)$$

also by assumption $\exists \beta_1, \beta_2 : I \to U$ with $[f] = [\alpha * \beta_1]$ and $[g] = [\alpha * \beta_2]$. This means that f and g have same initial & final points so we can hope that $[f] = [g]$. Investigate further, note that $[\beta_1 * \tilde{\beta}_2] = [e_x]$



U a contractible chart domain $\Rightarrow$ can shrink loop to point.

thus we consider the following

$$[\alpha] = [\alpha * e_x]$$

$$\Rightarrow [\alpha] = [\alpha * \beta_1 * \tilde{\beta}_2]$$

$$\Rightarrow [\alpha * \beta_2] = [\alpha * \beta_1 * \tilde{\beta}_2 * \beta_2] = [\alpha * \beta_1]$$

$$\Rightarrow [g] = [f] \quad \therefore \quad \boxed{P|_{U_{[\alpha]}} \text{ is injective}}$$

PROBLEM TWO — COVERING MANIFOLDS II

<u>Claim</u>: $p^{-1}(U) = \bigcup_{\substack{[\alpha] \\ \alpha(1)=x \\ \alpha(0)=x_0}} U_{[\alpha]}$

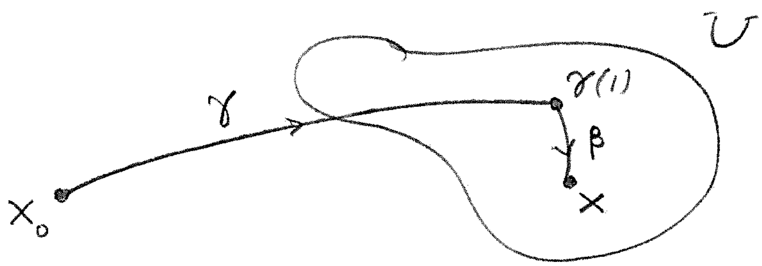
<u>Proof</u>: to begin we may use part (a).

$$P(([\alpha], U)) = U \Rightarrow ([\alpha], U) \subseteq P^{-1}(U)$$
$$\Rightarrow \bigcup_{\substack{[\alpha] \\ \alpha(1)=x \\ \alpha(0)=x_0}} ([\alpha], U) \subseteq P^{-1}(U).$$

Now suppose $[\gamma] \in P^{-1}(U)$, we seek to show $[\gamma] \in \bigcup_{\substack{[\alpha] \\ \alpha(1)=x \\ \alpha(0)=x_0}} ([\alpha], U)$.

Since $[\gamma] \in P^{-1}(U) \Rightarrow P([\gamma]) = \gamma(1) \in U$.

But $\gamma(1) \neq x$ necessarily, however since U is path connected $\exists$ a path $\beta : I \to U$ beginning at $\gamma(1)$ and ending at x. Note $(\gamma * \beta)(1) = x$,



we choose $[\alpha] = [\gamma * \beta]$ and argue $[\gamma] \in ([\alpha], U)$ since

$$[\gamma] = [\gamma * \beta * \tilde{\beta}] = [\alpha * \tilde{\beta}]$$

note $\tilde{\beta}$ is a path in U $\therefore$ $[\gamma] \in ([\alpha], U)$

$\therefore$ $[\gamma] \in \bigcup_{\substack{[\alpha] \\ \alpha(1)=x \\ \alpha(0)=x_0}} ([\alpha], U)$

Hence $P^{-1}(U) \subseteq \bigcup_{\substack{[\alpha] \\ \alpha(0)=x_0,\ \alpha(1)=x}} ([\alpha], U)$ $\therefore$ $\boxed{P^{-1}(U) = \bigcup_{\substack{[\alpha] \\ \alpha(0)=x_0,\ \alpha(1)=x}} ([\alpha], U)}$

Take - Home Problems

JAMES COON

PROBLEM 1 Let M be a connected manifold and fix $x_0 \in M$. Let $\tilde{M}$ denote the set of all homotopy classes of paths $\alpha : [0,1] \longrightarrow M$ such that $\alpha(0) = x_0$. Define a mapping $p : \tilde{M} \longrightarrow M$ by $p([\alpha]) = \alpha(1)$.

Show p is surjective

Recall that M a connected manifold $\Rightarrow$ M is pathwise connected.

Let $y \in M$ then $\exists\, \alpha : I \longrightarrow M$ with $\alpha(0) = x_0$ and $\alpha(1) = y$.

Observe $p([\alpha]) = \alpha(1) = y$ $\therefore$ p is surjective.

PROBLEM TWO REWORKED

We defined $U_{[\alpha]} = ([\alpha], U)$ for open U. Let

$$B_1 \equiv ([\alpha_1], U_1)$$
$$B_2 \equiv ([\alpha_2], U_2)$$

such that $B_1 \cap B_2 \neq \emptyset$. Let $[\gamma] \in B_1 \cap B_2$ we claim that,

$$B_3 = ([\gamma], U_1 \cap U_2)$$

is a subset of $B_1 \cap B_2$ containing $[\gamma]$. Let $[f] \in B_3$ then $\exists\, \delta : I \to U_1 \cap U_2$ where $[f] = [\gamma * \delta]$. We wish to show $[f] \in B_1$ and $[f] \in B_2$, towards that end notice $[\gamma] \in B_1 \cap B_2 \Rightarrow \exists\, \beta_1 : I \to U_1$ & $\beta_2 : I \to U_2$

$$[\gamma] = [\alpha_1 * \beta_1]$$
$$[\gamma] = [\alpha_2 * \beta_2]$$

thus it follows, from $[f] = [\gamma * \delta]$ that

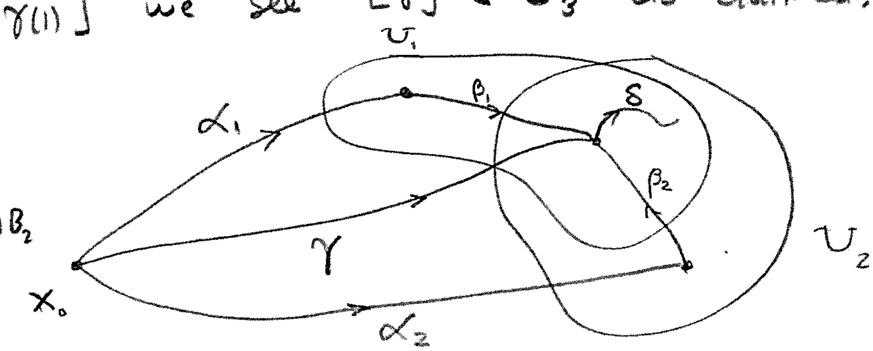
$$[f] = [\alpha_1 * \beta_1 * \delta]$$
$$[f] = [\alpha_2 * \beta_2 * \delta]$$

then since $\beta_1 * \delta$ is a path in U_1 and $\beta_2 * \delta$ is a path in U_2 we find $[f] \in B_1$ and $[f] \in B_2$ thus

$$[f] \in B_1 \cap B_2 \Rightarrow B_3 \subset B_1 \cap B_2$$

Since $e_{\gamma(1)}(I) = \gamma(1)$ is a path in $U_1 \cap U_2$ and $[\gamma] = [\gamma * e_{\gamma(1)}]$ we see $[\gamma] \in B_3$ as claimed.

Thus for all $[\gamma] \in B_1 \cap B_2$ $\exists$ another basic open set $B_3 \subset B_1 \cap B_2$ with $[\gamma] \in B_3$.



Off to a wrong start on this part.

PROBLEM TWO

For each open set $U \subseteq M$ and each path α in M such that $\alpha(1) \in U$ we define a set in $\tilde{M}$ as follows,

$$U_{[\alpha]} = \{ [\alpha * \beta] \mid \beta : I \to U \text{ and } \alpha(1) = \beta(0) \} = (U, [\alpha])$$

We claim $\mathcal{B} = \{\emptyset\} \cup \mathcal{U}$ is a basis for a topology on $\tilde{M}$ where $\mathcal{U} \equiv \{ U_{[\alpha]} \mid U \subset M \text{ and } \alpha : I \to M \text{ a path with } \alpha(1) \in U, \ \alpha(0) = x_0 \}$.

It suffices to show, if $B_1, B_2 \in \mathcal{B}$ with $B_1 \cap B_2 \neq \emptyset$ then there exists $B_3 \subset B_1 \cap B_2$. So suppose $B_1, B_2 \in \mathcal{B}$ with $B_1 \cap B_2 \neq \emptyset$

Not quite, need to show $\forall [\gamma] \in B_1 \cap B_2 \ \exists B_3 \ni [\gamma] \in B_3 \subseteq B_1 \cap B_2$

$$[\gamma] \begin{cases} B_1 = (U_1, [\alpha_1]) \\ B_2 = (U_2, [\alpha_2]) \end{cases}$$

should be $[\gamma]$ — B_1, B_2 are sets of classes of paths

Since $B_1 \cap B_2 \neq \emptyset$ $\exists y \in B_1 \cap B_2$. Also $\exists \beta_1 : I \to U_1$ and $\beta_2 : I \to U_2$ such that $\beta_1(1) = \beta_2(1) = y$.

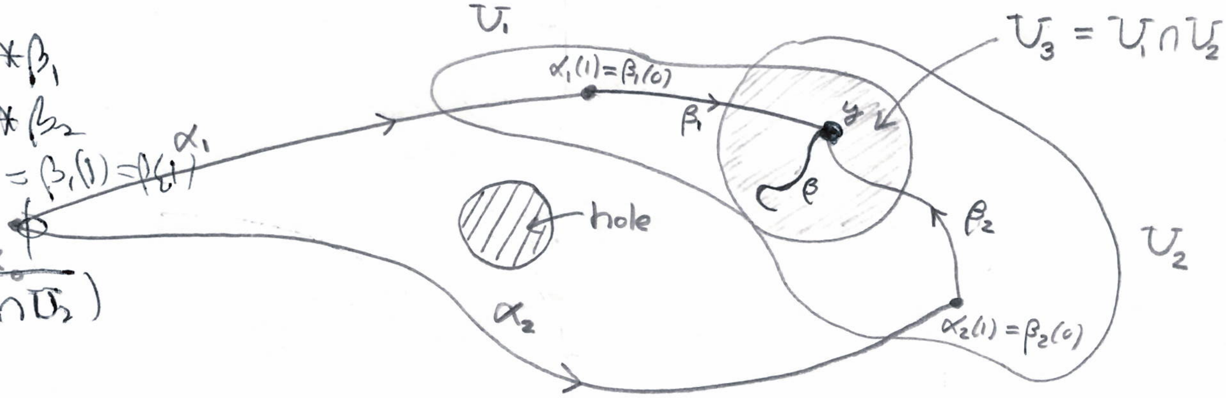
$\gamma \simeq \alpha_1 * \beta_1$

$\gamma \simeq \alpha_2 * \beta_2$

$y = \gamma(1) = \beta_1(1) = \beta_2(1)$

$U_1 \cap U_2 \neq \emptyset$

$(U_1 \cap U_2)$



We denote the path β_1 reversed by $\tilde{\beta}_1(s) = \beta_1(1-s)$. Notice we may relable B_1 as

$$B_1 = (U_1, [\alpha_2 * \beta_2 * \tilde{\beta}_1])$$

since $(\alpha_2 * \beta_2 * \tilde{\beta}_1)(1) = \alpha_1(1)$, where "$*$" is the concatenation of paths operation.

We define $B_3 \equiv (U_1 \cap U_2, [\alpha_2 * \beta_2])$ notice this is in $\mathcal{U}$ since $\alpha_2 * \beta_2$ is a path from x_0 to $y \in U_1 \cap U_2$.

Next we show $B_3 \subset B_1 \cap B_2$.

B_1, B_2

$B_1 \cap B_2 = \bigcup_\alpha B_\alpha$

PROBLEM TWO CONTINUED

Let $[f] \in B_3 \Rightarrow \exists \beta : I \to U_1 \cap U_2$ with $\beta(0) = (\alpha_2 * \beta_2)(1) = y$ s.t. $[f] = [\alpha_2 * \beta_2 * \beta]$. Clearly $[f] \in B_2$ since $\beta_2 * \beta$ is a path in U_2 since β_2 is a path in U_2 and β is also a path in U_2. Next, observe

$$\begin{aligned}[f] &= [\alpha_2 * \beta_2 * \beta] \\ &= [\alpha_2 * \beta_2 * \tilde{\beta}_1 * \beta_1 * \beta] \\ &= [(\alpha_2 * \beta_2 * \tilde{\beta}_1) * \underbrace{(\beta_1 * \beta)}_{\text{a path in } U_1}]\end{aligned}$$

$$\Rightarrow [f] \in B_1 = (U_1, [\alpha_2 * \beta_2 * \tilde{\beta}_1])$$

Thus $[f] \in B_1$ and $[f] \in B_2$ $\therefore$ $[f] \in B_1 \cap B_2$ & so $B_3 \subset B_1 \cap B_2$.

Finally, notice the trivial path $\gamma : I \to U_1 \cap U_2$ defined by $\gamma(s) = y$ gives $[\alpha_2 * \beta_2 * \gamma] \in B_3$ since γ is obviously a path in U_3, thus $B_3 \neq \emptyset$.

Hence we find $\mathcal{B} = \mathcal{U} \cup \{\emptyset\}$ forms a topological basis for $\tilde{M}$ and $U_{[\alpha]}$ are the basic open sets.

PROBLEM FOUR

Let U be open in $\mathcal{M}$. Let $x \in P(U_{[\alpha]}) \subseteq \mathcal{M}$, Consider

$$W \equiv \{ y \in U \mid \exists \beta : I \to U \ \& \ \beta(0) = x, \ \beta(1) = y \}$$

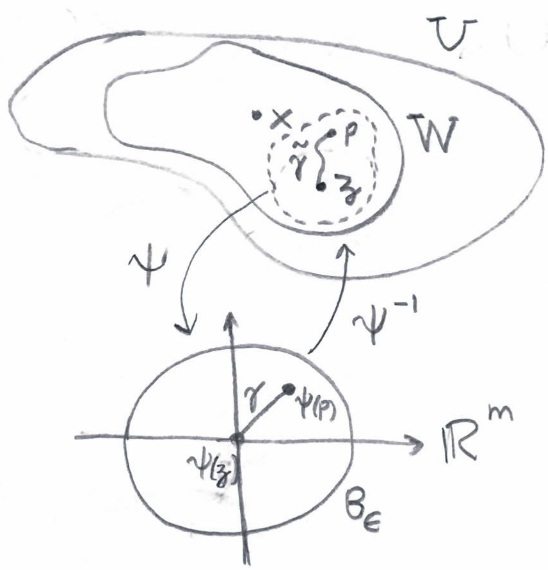
We note $P(U_{[\alpha]}) \subset U$. We seek to show W open. Let us take an arbitrary point $z \in W$. Since U is an open subset of a manifold $\mathcal{M}$ $\Rightarrow$ we can find a chart ψ centered at z such that $\psi(z) = 0$ and an open ball about zero in $\mathbb{R}^m$ say $B_\epsilon \subset \mathbb{R}^m$ such that $\psi^{-1}(B_\epsilon) \subset U$ (We assume the topology on $\mathcal{M}$ is induced from the metric topology on $\mathbb{R}^m$ by the bijective charts). We wish to show that $\psi^{-1}(B_\epsilon) \subset W$ as that will estabish that each point $z \in W$ has an open nbhd about it contained in W hence W will be open.

Let $p \in \psi^{-1}(B_\epsilon)$

then $\psi(p) \in B_\epsilon$,

note we have a path from $\psi(z)$ to $\psi(p)$, namely

$$\gamma(t) = t\psi(p)$$
$$\gamma(0) = 0 = \psi(z).$$
$$\gamma(1) = \psi(p)$$



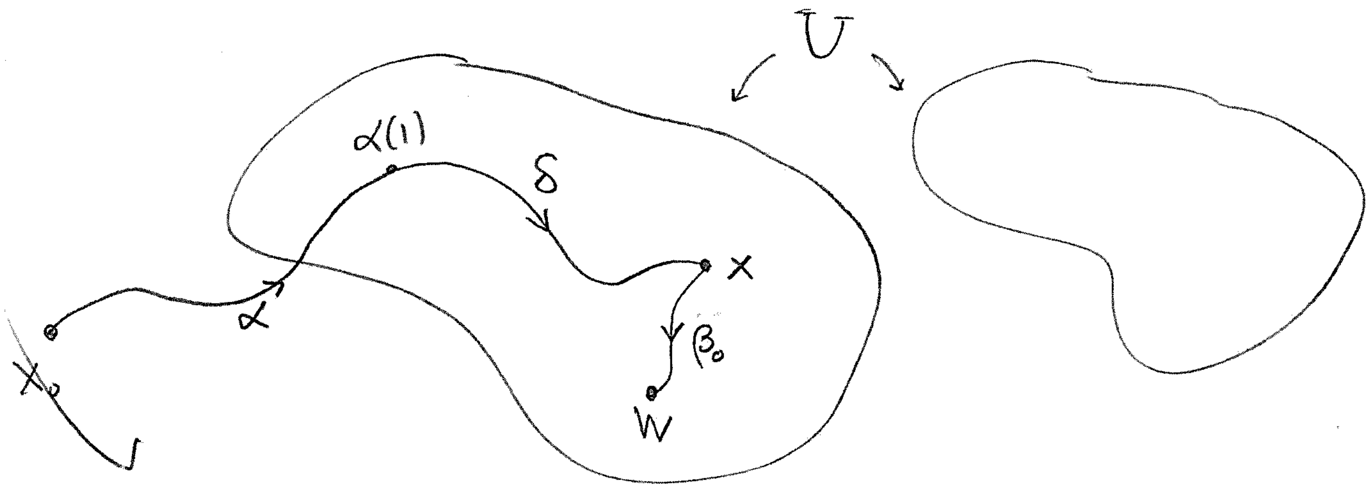
Also note $\|\gamma(t)\| = \|t\psi(p)\| = |t| \ \|\psi(p)\| < 1 \cdot \epsilon = \epsilon$ for each $t \in [0,1]$ hence $\gamma(I) \subset B_\epsilon$. We can lift γ to $\tilde{\gamma} : I \to M$ where $\tilde{\gamma} \equiv \psi^{-1} \circ \gamma$, this is a path from $\tilde{\gamma}(0) = \psi^{-1}(\gamma(0)) = \psi^{-1}(\psi(z)) = z$. to $\tilde{\gamma}(1) = \psi^{-1}(\gamma(1)) = \psi^{-1}(\psi(p)) = p$. Since $z \in W$ we have β from x to z and so $\beta * \tilde{\gamma}$ is a path from x to p thus $p \in W$. $\Rightarrow \psi^{-1}(B_\epsilon) \subset W$.

PROBLEM FOUR CONTINUED

We just saw that for each $x \in P(U_{[\alpha]}) \subset U$ $\exists\ W$ open about x, we argue now that $W \subset P(U_{[\alpha]})$. Let $w \in W$ then $\exists$ a path β_0 from x to w by the def^{n} of W.

Note $x \in P((U, [\alpha])) \Rightarrow \exists$ a path δ in U such that

$$P([\alpha * \delta]) = (\alpha * \delta)(1) = x$$



thus $(\alpha * \delta * \tilde{\beta}_0)(1) = w$ & $\delta * \tilde{\beta}_0$ is a path in U thus $w \in P(U_{[\alpha]})$ since $[\alpha * \delta * \tilde{\beta}_0] \in U_{[\alpha]}$ and $P([\alpha * \delta * \tilde{\beta}_0]) = (\alpha * \delta * \tilde{\beta}_0)(1) = w$. Hence $W \subset P(U_{[\alpha]})$ $\therefore$ $P(U_{[\alpha]})$ is open in M. Since $\mathcal{U}$ is a topological basis $\Rightarrow$ P is an open map as claimed.