66/66 (1)

HWK # 3, 5, 8 assigned 9/13/06

(#3.) Check properties 2.12 of the Lie Product

The "Lie product" $LG \times LG \longrightarrow LG$ where $(X, Y) \longmapsto [X, Y]$ is bilinear and hence makes LG a $\mathbb{R}$-algebra also

(i.) $[X, X] = 0$ hence $[X, Y] = -[Y, X]$ (Skew)

(ii) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ (Jacobi)

$LG \equiv$ LIVF on G & the bracket is the usual vector field bracket $[X, Y]f = X(Y(f)) - Y(X(f))$

Proof: Bilinearity: $[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z]$ we find

Let $f \in C^\infty(G)$ then for $X, Y, Z \in LG$

$$\begin{aligned}
[\alpha X + \beta Y, Z]f &= (\alpha X + \beta Y)(Z(f)) - Z((\alpha X + \beta Y)(f)) \\
&= \alpha X(Z(f)) + \beta Y(Z(f)) - Z(\alpha X(f) + \beta Y(f)) \\
&= \alpha\big(X(Z(f)) - Z(X(f))\big) + \beta\big(Y(Z(f)) - Z(Y(f))\big) \\
&= \alpha[X, Z]f + \beta[Y, Z]f \\
&= (\alpha[X, Z] + \beta[Y, Z])f \quad \forall f \in C^\infty(G)
\end{aligned}$$

$\therefore [\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z]. \quad \forall \alpha, \beta \in \mathbb{R}.$

Likewise $[X, \alpha Y + \beta Z] = \alpha[X, Y] + \beta[X, Z] \quad \forall \alpha, \beta \in \mathbb{R}$ & $X, Y, Z \in LG$.

Skew: $[X, X]f = X(X(f)) - X(X(f)) = 0(f)$

$\therefore [X, X] = 0 \quad \forall X \in LG$. Then $X, Y \in LG$

$\Rightarrow X - Y \in LG$ as we proved in lecture thus

$$\begin{aligned}
&[X - Y, X - Y] = 0 \\
&\Rightarrow [X, X - Y] + [-Y, X - Y] = 0 \\
&\Rightarrow [X, X] - [X, Y] - [Y, X] + [Y, Y] = 0 \\
&\Rightarrow [X, Y] = -[Y, X] \quad \forall X, Y \in LG.
\end{aligned}$$

Jacobi: $\big([[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y]\big)f \overset{?}{=} 0$

$[X, Y]Z(f) - Z([X, Y]f) + [Y, Z]X(f) - X([Y, Z]f) + [Z, X]Y(f) - Y([Z, X]f) = 0$

$$\begin{aligned}
&\cancel{X(Y(Z(f)))}_{1} - \cancel{Y(X(Z(f)))}_{2} - \cancel{Z(X(Y(f)) - Y(X(f)))}_{3,4} \\
&\cancel{Y(Z(X(f)))}_{6} - \cancel{Z(Y(X(f)))}_{4} - \cancel{X(Y(Z(f)) - Z(Y(f)))}_{1,5} \\
&\cancel{Z(X(Y(f)))}_{3} - \cancel{X(Z(Y(f)))}_{5} - \cancel{Y(Z(X(f)) - X(Z(f)))}_{6,2}
\end{aligned}$$

Jacobi Holds.

(#5) Check that the Lie algebras of $SO(n)$, $U(n)$, $SL(n,\mathbb{R})$ $SL(n,\mathbb{C})$, and $Sp(n)$ are closed under the Lie product of matrices

$L(SO(n)) = so(n) = \{a \in gl(n,\mathbb{R}) \mid a^T = -a\}$. Let $a, b \in so(n)$

$$[a,b]^T = (ab)^T - (ba)^T = b^T a^T - a^T b^T$$
$$= -b(-a) - (-a)(-b)$$
$$= -(ab - ba)$$
$$= -[a,b] \therefore [a,b] \in so(n).$$

Let $C \in SO(n)$ then $C^T C = I$ thus $C^T = C^{-1}$ then let $a \in so(n)$ and consider,

$$(C a C^{-1})^T = (C^{-1})^T a^T C^T$$
$$= C^{TT}(-a) C^{-1}$$
$$= -C a C^{-1} \quad \therefore \quad C a C^{-1} \in so(n)$$

$G_J = \{A \mid A^\dagger J A = J\}$ (or T for $\mathbb{R}$)
$\mathfrak{g}_J = \{a \mid a^\dagger J + JA = 0\}$ (or T for $\mathbb{R}$)

$$[a,b]^\dagger J + J[a,b] = ((ab)^\dagger - (ba)^\dagger)J + J(ab - ba)$$
$$= (b^\dagger a^\dagger J - a^\dagger b^\dagger J + Jab - Jba)$$
$$= (b^\dagger(-JA) - a^\dagger(-Jb) + Jab - Jba$$
$$= \cancel{Jba} - \cancel{Jab} + \cancel{Jab} - \cancel{Jba}$$
$$= 0 \quad \therefore \quad [a,b] \in \mathfrak{g}_J$$

OK but ~~his~~ sp is based on ℍ and is a little different

derivation works for T or $\dagger$.

Let $C \in G_J$ so $C^\dagger J C = J \Rightarrow C^\dagger J = J C^{-1}$. Let $a \in \mathfrak{g}_J$ and consider that $JC = (C^\dagger)^{-1} J$ as well so,

$$(C a C^{-1})^\dagger J + J(C a C^{-1}) = (C^{-1})^\dagger a^\dagger C^\dagger J + J C a C^{-1}$$
$$= (C^{-1})^\dagger a^\dagger J C^{-1} + J C a C^{-1}$$
$$= -(C^{-1})^\dagger J a C^{-1} + J C a C^{-1}$$
$$= -J \cancel{C a C^{-1}} + J \cancel{C a C^{-1}}$$
$$= 0 \quad \therefore \quad C a C^{-1} \in \mathfrak{g}_J$$

This covers all the cases listed just choose appropriate J and either T or $\dagger$, Except the special linear cases pg. 22 #5

- Check that $\mathfrak{sl}(n,\mathbb{R})$ & $\mathfrak{sl}(n,\mathbb{C})$ are closed under the commutator bracket & conjugation via their corresponding Lie groups $SL(n,\mathbb{R})$ & $SL(n,\mathbb{C})$. In the arguments below $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

$$\mathfrak{sl}(n,\mathbb{K}) = \{ a \in \mathfrak{gl}(n,\mathbb{K}) \mid tr(a) = 0 \}$$

Let $a, b \in \mathfrak{sl}(n,\mathbb{K})$ then

$$\begin{aligned} tr[a,b] &= tr(ab - ba) \\ &= tr(ab) - tr(ba) \\ &= tr(ab) - tr(ab) \\ &= 0 \quad \therefore \ \mathfrak{sl}(n,\mathbb{K}) \text{ is closed under } [\,,\,]. \end{aligned}$$

Next recall $SL(n,\mathbb{K}) = \{ A \in \mathfrak{gl}(n,\mathbb{K}) \mid det(A) = 1 \}$
Let $C \in SL(n,\mathbb{K})$ then $det(C) = 1 \therefore C^{-1}$ exists &, let $a \in \mathfrak{sl}(n,\mathbb{K})$ then

$$\begin{aligned} trace(C a C^{-1}) &= tr(C^{-1} C a) \\ &= tr(a) \\ &= 0 \quad \therefore \ C a C^{-1} \in \mathfrak{sl}(n,\mathbb{K}). \end{aligned}$$

Remark: $\mathfrak{so}(n)$ is invariant under conjugation by $O(n)$ as well, we never used $det(C) = 1$ in that earlier argument.

$\boxed{\#8 \text{ pg. } 22}$ We define $so(3) = \{ A \in gl(n, \mathbb{R}) \mid A^T = -A \}$. We can write an arbitrary $A \in so(3)$ via $A_1, A_2, A_3 \in \mathbb{R}$,

$$A = \begin{bmatrix} 0 & -A_3 & A_2 \\ A_3 & 0 & -A_1 \\ -A_2 & A_1 & 0 \end{bmatrix} = A_i \epsilon_{ijk} E_{kj}$$

where $i, j, k = 1, 2, 3$ and we mean to use repeated index convention so Σ's are implied, also $\epsilon_{123} = 1$ and ϵ_{ijk} is completely antisymmetric. Let $A, B \in so(3)$,

$$\begin{aligned}
[A, B] &= [A_i \epsilon_{ijk} E_{kj} , B_l \epsilon_{lmn} E_{nm}] \\
&= A_i B_l \epsilon_{ijk} \epsilon_{lmn} [E_{kj}, E_{nm}] \quad \leftarrow \text{This bracket is defined on } gl(3) \text{ since } E_{ij} \notin so(3), \ so(3) \text{ bracket is same as } gl(3) \text{ bracket.} \\
&= A_i B_l \epsilon_{ijk} \epsilon_{lmn} (\delta_{jn} E_{km} - \delta_{mk} E_{nj}) \\
&= A_i B_l \epsilon_{ijk} \epsilon_{lmj} E_{km} - A_i B_l \epsilon_{ijk} \epsilon_{lkn} E_{nj} \\
&= A_i B_l (-\epsilon_{ikj} \epsilon_{lmj}) E_{km} - A_i B_l (-\epsilon_{ijk} \epsilon_{lnk}) E_{nj} \\
&= A_i B_l (-\delta_{il} \delta_{km} + \delta_{kl} \delta_{im}) E_{km} - A_i B_l (-\delta_{il} \delta_{jn} + \delta_{jl} \delta_{in}) E_{nj} \\
&= \cancel{-A_i B_i E_{kk}} + A_m B_k E_{km} + \cancel{A_l B_l E_{nn}} - A_n B_j E_{nj} \\
&= (A_i B_j - A_j B_i) E_{ji} = A_i B_j (E_{ji} - E_{ij})
\end{aligned}$$

<u>Lemma (I)</u>: $[A, B] = A_i B_j (E_{ji} - E_{ij})$. Proof above

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We define $\varphi : so(3) \longrightarrow \mathbb{R}^3$ on $A = A_i \epsilon_{ijk} E_{kj} \in so(3)$ by

$$\varphi(A) \equiv A_i e_i$$

We show φ is linear. Let $A, B \in so(3)$ and let $c \in \mathbb{R}$ as usual $A = A_i \epsilon_{ijk} E_{kj}$ and $B = B_l \epsilon_{lmn} E_{nm}$. Note,

$$\begin{aligned}
\varphi(A + cB) &= \varphi(A_i \epsilon_{ijk} E_{kj} + cB_l \epsilon_{lmn} E_{nm}) \\
&= \varphi((A_i + cB_i)\epsilon_{ijk} E_{kj}) \\
&= (A_i + cB_i) e_i \\
&= A_i e_i + c(B_i e_i) \\
&= \varphi(A_i \epsilon_{ijk} E_{kj}) + c\varphi(B_i \epsilon_{ijk} E_{kj}) \\
&= \varphi(A) + c\varphi(B).
\end{aligned}$$

We observe that $\{\epsilon_{ijk} E_{kj}\}_{i=1}^{3} = \{E_{32} - E_{23}, E_{13} - E_{31}, E_{21} - E_{12}\}$ is a basis for $so(3)$ as they are clearly LI and span as we have pointed out before $A \in so(3) \Rightarrow A = A_i \epsilon_{ijk} E_{kj}$. Notice that for $m = 1, 2, 3$ we have

$$\begin{aligned}
\varphi(\epsilon_{mjk} E_{kj}) &= \varphi(\delta_{mi} \epsilon_{ijk} E_{kj}) \\
&= \delta_{mi} e_i \qquad (\text{Here } A_i = \delta_{mi}) \\
&= e_m,
\end{aligned}$$

Thus φ maps the basis of $so(3)$ to the basis of $\mathbb{R}^3$ $\therefore$ $\underline{\varphi \text{ is a linear isomorphism}}$.

$\boxed{\underline{\text{Lemma II}}: \ \varphi(E_{nm} - E_{mn}) = \epsilon_{mni}\, e_i}$

__Proof__: By definition $\varphi(\epsilon_{ijk} E_{kj}) = e_i$. Now multiply (and sum) by ϵ_{mni} and recall φ is linear,

$$\begin{aligned}
\epsilon_{mni}\, \varphi(\epsilon_{ijk} E_{kj}) &= \varphi(\epsilon_{mni} \epsilon_{ijk} E_{kj}) \\
&= \varphi\big((\delta_{mj}\delta_{nk} - \delta_{nj}\delta_{mk}) E_{kj} \big) \\
&= \varphi(E_{nm} - E_{mn}) = \epsilon_{mni}\, e_i .
\end{aligned}$$

$\boxed{\underline{\text{Claim}}: \ \varphi([A,B]) = \varphi(A) \times \varphi(B). \ \text{ for } A, B \in so(3).}$

__Proof__: Begin by noting $\varphi(A) = A_i e_i$ and $\varphi(B) = B_j e_j$ so that $\varphi(A) \times \varphi(B) = A_i B_j \epsilon_{ijk} e_k$. Consider,

$$\begin{aligned}
\varphi([A,B]) &= \varphi\big(A_m B_n [E_{nm} - E_{mn}] \big) && \text{: by Lemma I.} \\
&= A_i B_j\, \varphi(E_{nm} - E_{mn}) && \text{: linearity of } \varphi. \\
&= A_i B_j\, \epsilon_{mnk}\, e_k && \text{: by Lemma II.} \\
&= \varphi(A) \times \varphi(B).
\end{aligned}$$

OK
notation
error

 (Working towards $\varphi(AXA^{-1}) = A\varphi(X) \quad \forall X \in so(3)$ & $A \in SO(3)$.

We begin by noting that one can calculate the determinant via $e^1A \wedge e^2A \wedge e^3A = \det(A)\, e^1 \wedge e^2 \wedge e^3$ where I'm using e^i as the i^{th} standard row-vector. Notice that $A = A_{mn}E_{mn}$ thus $e^j A_{mn}E_{mn} = A_{mn}e^j E_{mn}$ and since $e^j E_{mn} = \delta^j_m e^n \Rightarrow e^j A = A_{jn}e^n$ hence

$$\begin{aligned} e^1A \wedge e^2A \wedge e^3A &= A_{1i}e^i \wedge A_{2j}e^j \wedge A_{3k}e^k \\ &= A_{1i}A_{2j}A_{3k}\, e^i \wedge e^j \wedge e^k \\ &= A_{1i}A_{2j}A_{3k}\,\epsilon_{ijk}\, e^1 \wedge e^2 \wedge e^3 \\ &= \det(A)\, e^1 \wedge e^2 \wedge e^3 \end{aligned}$$

So for the disinterested reader we could just begin with,

$$\boxed{\det(A) = A_{1i}A_{2j}A_{3k}\,\epsilon_{ijk}}$$

This suggest we should rearrange the problem so there are three A's on the same side. In the following we use the fact φ is an isomorphism repeatedly, also for all $X \in so(3)$ and $A \in SO(3)$ we proved previously $AXA^{-1} \in so(3)$ so we may take φ of it,

$$\begin{aligned} \varphi(AXA^{-1}) = A\varphi(X) &\iff AXA^{-1} = \varphi^{-1}(A\varphi(X)) \\ &\iff X = A^{-1}\varphi^{-1}(A\varphi(X))A \end{aligned}$$

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$[A\phi(\mathbb{X})]_i = A_{ij}\,\phi(\mathbb{X})_j$, $\Phi(\mathbb{X})_j = \mathbb{X}_j$

$\therefore$ $[A\phi(\mathbb{X})]_i = A_{ij}\,\mathbb{X}_j$

· We collect a few results,

$$A\phi(x) = (A_{mn}X_n)e_m$$

$$\phi^{-1}(A\phi(x)) = A_{mn}X_n\,\phi^{-1}(e_m) = A_{mn}X_n\,\epsilon_{mjk}E_{kj}$$

$$\left(A^{-1}\phi^{-1}(A\phi(x))\right)_{ab} = (A^{-1})_{ap}\left(\phi^{-1}(A\phi(x))\right)_{pb}$$

$$= A_{pa}A_{mn}X_n\,\epsilon_{mbp} \quad , \text{ since } A^{-1} = A^T.$$

$$\left(A^{-1}\phi^{-1}(A\phi(x))A\right)_{az} = \left(A^{-1}\phi^{-1}(A\phi(x))\right)_{ab}A_{bz}$$

$$= A_{pa}A_{mn}X_n\,\epsilon_{mbp}A_{bz}$$

Thus we have reduced the question of $\varphi(AXA^{-1}) \overset{?}{=} A\varphi(x)$ to the following component form of $X \overset{?}{=} A^{-1}\varphi^{-1}(A\varphi(x))A$ where we consider the $(az)^{th}$ component of that matrix eq^{n}, notice $(X)_{az} = (X_\mu\,\epsilon_{\mu\alpha\beta}E_{\beta\alpha})_{az} = X_\mu\,\epsilon_{\mu za}$ hence, $\phi(\mathbb{X}) = (x_\mu)$

$$\boxed{\epsilon_{mbp}A_{mn}A_{bz}A_{pa}X_n \overset{?}{=} X_\mu\,\epsilon_{\mu za}}$$

We claim this is true due to the fact $A \in SO(3)$ $\Rightarrow \det(A) = 1$ $\therefore$ $\varphi(AXA^{-1}) = A\varphi(x)$ $\forall x \in so(3)$ & $A \in SO(3)$.

We prove this claim on next page.

Claim: $\epsilon_{mbp} A_{mn} A_{bz} A_{pa} X_n = X_\mu \epsilon_{\mu z a}$ where A_{mn} are the components of $A \in SO(3)$.

Proof: Notice $\{z, a\}$ are the free indices, we must prove the eq^n holds for all choices of $\{z, a\} \subset \{1,2,3\}$

CASE I: Suppose $z = a$, no sum over repeated z below,

RHS : $X_\mu \epsilon_{\mu z z} = 0.$

LHS :
$$\begin{aligned}
\epsilon_{mbp} A_{mn} A_{bz} A_{pz} &= -\epsilon_{mpb} A_{mn} A_{bz} A_{pz} \quad : \epsilon_{mbp} = -\epsilon_{mpb} \\
&= -\epsilon_{mpb} A_{mn} A_{pz} A_{bz} \\
&= -\epsilon_{mbp} A_{mn} A_{bz} A_{pz} \quad : \text{switched } (p \leftrightarrow b) \\
&= 0.
\end{aligned}$$

CASE II: Suppose $z \neq a$,

RHS : $X_\mu \epsilon_{\mu z a} = \epsilon_{\nu z a} X_\nu$, no sum over ν (fixed but arbitrary) and $\{\nu, z, a\} = \{1,2,3\}$

LHS : $\epsilon_{mbp} A_{mn} A_{bz} A_{pa} X_n = \underbrace{\epsilon_{mbp} A_{m\nu} A_{bz} A_{pa} X_\nu}$

must be ν which is the remaining index value not taken by z or a. Other terms will be zero because either z or a will be repeated in the n-slot, this will make $A_{mn} A_{bz} A_{pa}$ symmetric in either mb or mp yet this is summed against antisymmetric in mp & mb $\epsilon_{mbp} \Rightarrow$ zero.

only nontrivial term. again we have chosen ν so that $\{\nu, z, a\} = \{1,2,3\}$ no ordering intended.

Coefficient of x_z is $\epsilon_{mbp} \underbrace{A_{mz} A_{bz}}_{mb} A_{pa}$ = zero

Coefficient of x_a is also zero

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CASE II: Continued. Recall $\{\nu, z, a\} = \{1,2,3\}$ they are three distinct indices. Again no sum over ν intended,

$$\epsilon_{\nu z a} X_\nu \overset{(?)}{=} \epsilon_{mbp} A_{m\nu} A_{bz} A_{pa} X_\nu$$

$$X_1 = \epsilon_{123} X_1 = \epsilon_{mbp} A_{m1} A_{b2} A_{p3} X_1 = \det(A) X_1 = X_1 .$$

$$\begin{aligned} X_2 = \epsilon_{231} X_2 &= \epsilon_{mbp} A_{m2} A_{b3} A_{p1} X_2 \\ &= \epsilon_{pmb} A_{p1} A_{m2} A_{b3} X_2 \\ &= \det(A) X_2 = X_2 . \end{aligned}$$

$$\begin{aligned} X_3 = \epsilon_{312} X_3 &= \epsilon_{mbp} A_{m3} A_{b1} A_{p2} X_3 \\ &= \epsilon_{bpm} A_{b1} A_{p2} A_{m3} X_3 \\ &= \det(A) X_3 = X_3 . \end{aligned}$$

$$\begin{aligned} -X_3 = \epsilon_{321} X_3 &= \epsilon_{mbp} A_{m3} A_{b2} A_{p1} X_3 \\ &= -\epsilon_{pbm} A_{p1} A_{b2} A_{m3} X_3 \\ &= -\det(A) X_3 = -X_3 . \end{aligned}$$

$$\begin{aligned} -X_2 = \epsilon_{213} X_2 &= \epsilon_{mbp} A_{m2} A_{b1} A_{p3} X_2 \\ &= -\epsilon_{bmp} A_{b1} A_{m2} A_{p3} X_2 \\ &= -\det(A) X_2 = -X_2 . \end{aligned}$$

$$\begin{aligned} -X_1 = \epsilon_{132} X_1 &= \epsilon_{mbp} A_{m1} A_{b3} A_{p2} X_1 \\ &= -\epsilon_{mpb} A_{m1} A_{p2} A_{b3} X_1 \\ &= -\det(A) X_1 = -X_1 . \end{aligned}$$

$$\therefore \quad \epsilon_{\nu z a} X_\nu = \epsilon_{mbp} A_{m\nu} A_{bz} A_{pa} X_\nu$$ for all distinct $\{\nu, z, a\}$.

Hence the Claim is verified. //