(5) PROBLEM 6 of pg. 10 [BD], assume that $SO(n)$ and $U(n)$ are connected.

$U(n) \subset SO(2n)$

We reinterpret to mean $\psi(U(n)) \subset SO(2n)$

Let $A \in U(n)$ we seek to show $\psi(A) \in SO(2n)$

Consider

$$\psi(A)^T \psi(A) = \begin{bmatrix} \varphi(A_1^1)^T & \varphi(A_1^2)^T & \cdots & \varphi(A_1^n)^T \\ \varphi(A_2^1)^T & \varphi(A_2^2)^T & & \\ \vdots & & \ddots & \\ \varphi(A_n^1)^T & & & \varphi(A_n^n)^T \end{bmatrix} \begin{bmatrix} \varphi(A_1^1) & \varphi(A_2^1) & \cdots & \varphi(A_n^1) \\ \varphi(A_1^2) & \varphi(A_2^2) & & \vdots \\ & & \ddots & \\ \varphi(A_1^n) & \cdots & & \varphi(A_n^n) \end{bmatrix}$$

We look at an arbitrary (2×2) block, the $\binom{p}{m}^{th}$ block,

$$\sum_{k=1}^{n} \varphi(A_p^k)^T \varphi(A_m^k) = \sum_{k=1}^{n} \varphi((A_p^k)^*)\varphi(A_m^k), \quad \text{since } \varphi(z)^T = \varphi(z^*).$$

$$= \sum_{k=1}^{n} \varphi((A_p^k)^* A_m^k), \quad \text{since } \varphi \text{ is homomorphism.}$$

$$= \sum_{k=1}^{n} \varphi(((A^T)^*)_k^p A_m^k), \quad (A^T)_j^i \equiv A_i^j$$

$$= \sum_{k=1}^{n} \varphi((A^\dagger A)_m^p)$$

$$= \sum_{k=1}^{n} \varphi(\delta_m^p) \quad \left(\text{since } A^\dagger A = I \Rightarrow (A^\dagger A)_m^p = \delta_m^p\right)$$

$$= \delta_m^p I_2.$$

Note: Can prove $\psi(A^\dagger) = \psi(A)^t$ and use $\psi(A^\dagger A) = \psi(A)^t \psi(A)$

$\therefore \ \psi(A)^T \psi(A) = I \quad \therefore \ \psi(A) \in O(2n)$.

$\Psi(U(n)) \subset SO(2n)$ Continued

We saw that $\Psi(U(n)) \subset O(2n)$. Now Ψ is a continuous map and we assumed that $U(n)$ is connected $\Rightarrow \Psi(U(n))$ is connected. Furthermore Ψ is a homomorphism thus $\Psi(e_{U(n)}) = e_{O(n)}$ so we know that $\Psi(U(n))$ contains the identity. This means $\Psi(A) \in SO(2n)$ since

$$O(2n) = \{ B \in gl(2n) \mid B^T B = I \}$$

then $\det(B^T B) = \det(B^T)\det(B) = (\det(B))^2 = 1$ these are real matrices $\therefore \det(B) = \pm 1$ so $O(2n)$ has two components, this is a seperation,

$$O(2n) = \underbrace{\{ B \in O(2n) \mid \det(B) = 1 \}}_{SO(2n)} \cup \{ B \in O(2n) \mid \det B = -1 \}$$

Again since $\Psi(U(n)) \subset O(2n)$ is connected & contains the identity $\Rightarrow \Psi(A) \in SO(n)$. $\therefore$ $\boxed{\Psi(U(n)) \subset SO(2n)}$

$\psi(U(n)) \subset SO(2n)$ Alternative viewpoint

$U(n)$ connected $\Rightarrow \exists \mathcal{B} \subset gl(n, \mathbb{C})$ s.t. $\exp(\mathcal{B}) = U(n)$.

Let $B \in \mathcal{B}$ determine what condition we must place on this generator of the group, $e^B \in U(n)$ thus

$$(e^B)^\dagger e^B = I \iff e^{B^\dagger} = e^{-B}$$
$$\iff B^\dagger = -B$$

Apparently $\mathcal{B}$ is the set of skew Hermitian matrices. Consider then the image of $\mathcal{B}$ under ψ, what can we say about those matrices,

$$B^\dagger = -B \Rightarrow B^* = -B^T$$
$$\Rightarrow \psi(B^*) = -\psi(B^T)$$
$$\Rightarrow \psi(B)^T = -\psi(B^T)$$

How do you get this?

$\psi(B^*) = \psi((B^*)^T)^T = \psi(B^*)^T$

need 2 transposes.

$$\Rightarrow (\varphi(B_p^m))^T = (-\varphi((B^T)_p^m))$$
$$\Rightarrow \varphi(B_m^p)^T = -\varphi(B_m^p)$$
$$\Rightarrow \varphi(B_m^p) \text{ are skew-symmetric.}$$
$$\Rightarrow \text{trace}(\varphi(B_m^p)) = 0 \quad (\text{for all } 1 \le p, m \le n)$$

Then we may calculate directly

$$\det(\psi(A)) = \det(\psi(e^B))$$
$$= \det(e^{\psi(B)}) \quad \text{since } \psi(AB) = \psi(A)\psi(B).$$
$$= \exp(\text{trace}(\psi(B))) \quad (\text{proved previously.})$$
$$= \exp(0) = 1 \quad \therefore \underline{\psi(A) \in SO(2n)}.$$

$\psi(GL(n,\mathbb{C})) \subset GL^+(2n,\mathbb{R})$

Let $A \in GL(n,\mathbb{C})$ then show $\psi(A) \in GL^+(2n,\mathbb{R})$ meaning that $\det(\psi(A)) > 0$. Clearly

$$GL(2n,\mathbb{R}) = GL^+(2n,\mathbb{R}) \cup GL^-(2n,\mathbb{R})$$

Then its known $GL(n,\mathbb{C})$ is connected thus as ψ is continous and $I \in GL^+(2n,\mathbb{R})$ and $I \in \psi(GL(n,\mathbb{C}))$ since ψ is a homomorphism

$$\Rightarrow \underline{\psi(GL(n,\mathbb{C})) \subset GL^+(2n,\mathbb{R})}.$$

<u>Alternatively</u>

$\exp : \mathfrak{gl}(n,\mathbb{C}) \longrightarrow GL(n,\mathbb{C})$ its onto since $GL(n,\mathbb{C})$ is connected & exp maps to connected component of identity. $A \in GL(n,\mathbb{C}) \Rightarrow A = e^B$ for $B \in \mathfrak{gl}(n,\mathbb{C})$,

$$\begin{aligned}\det(\psi(A)) &= \det(\psi(e^B)) \\ &= \det(e^{\psi(B)}) \quad \psi\text{ - homomorphism.} \\ &\qquad\qquad\qquad \text{\& continuity!} \\ &= \exp(\operatorname{trace}(\psi(B))) > 0\end{aligned}$$

$$\Rightarrow \psi(A) \in GL^+(2n,\mathbb{R})$$

$$\Rightarrow \underline{\psi(GL(n,\mathbb{C})) \subset GL^+(2n,\mathbb{R})}$$

$\boxed{\Psi(GL(n,\mathbb{C})) \cap SO(2n) = \Psi(U(n))}$

We proved previously $\Psi(U(n)) \subset SO(2n)$ $\therefore$ $\Psi(U(n)) \subset \Psi(GL(n,\mathbb{C})) \cap SO(2n)$
The other direction is more interesting, let $A \in \Psi(GL(n,\mathbb{C})) \cap SO(2n)$
then we have

$$A \in SO(2n) \Rightarrow A^T A = I \;\;\&\;\; \det(A) = 1.$$

$$A \in \Psi(GL(n,\mathbb{C})) \Rightarrow \exists B \in GL(n,\mathbb{C}) \text{ such that } \Psi(B) = A.$$

It is useful to notice

$$\Psi(B)^T = \begin{bmatrix} \varphi(B_1^1)^T & \varphi(B_1^2)^T & \cdots & \varphi(B_1^n)^T \\ \varphi(B_2^1)^T & \varphi(B_2^2)^T & & \vdots \\ \vdots & & \ddots & \\ \varphi(B_n^1)^T & \cdots & & \varphi(B_n^n)^T \end{bmatrix}$$

$$= \begin{bmatrix} \varphi((B_1^1)^*) & \varphi((B_1^2)^*) & \cdots & \varphi((B_1^n)^*) \\ \vdots & & & \\ \varphi((B_n^1)^*) & & \cdots & \varphi((B_n^n)^*) \end{bmatrix}$$

$$= \Psi((B^T)^*)$$

$$= \Psi(B^\dagger)$$

Thus,

$$I_{2n} = A^T A = \Psi(B)^T \Psi(B)$$
$$= \Psi(B^\dagger)\Psi(B)$$
$$= \Psi(B^\dagger B) \Rightarrow B^\dagger B = \Psi^{-1}(I_{2n}) = I_n$$

$\therefore$ $B \in U(n)$

thus $\Psi(B) = A \Rightarrow A \in \Psi(U(n))$.

$\Rightarrow GL(n,\mathbb{C}) \cap SO(2n) \subset \Psi(U(n))$

$\therefore$ $\boxed{GL(n,\mathbb{C}) \cap SO(2n) = \Psi(U(n))}$

$\underline{GL(2n,\mathbb{R}) \cap \psi(U(n)) = O(2n)}$

$\bigcirc 10$

$\psi(U(n)) \subset SO(2n) \subset O(2n)$ from previous work.

Let $A \in O(2n)$ then $A^T A = I$.

1.) $\det(A^T A) = \det(A^T)\det(A) = \det(A)^2 = \det(I) = 1$

$\therefore \det(A) \neq 0 \Rightarrow A \in GL(2n,\mathbb{R})$.

Why is A in the image of ψ?

$\psi^{-1}(A)$ may not be defined

2.) Let $B = \psi^{-1}(A)$ then $\psi(B) = A$ and,

$$
\begin{aligned}
\psi(B^\dagger B) &= \psi(B^\dagger)\psi(B) \\
&= \psi(B)^T \psi(B) \\
&= A^T A \\
&= I \quad \Rightarrow B^\dagger B = I \\
& \quad \Rightarrow B \in U(n) \\
& \quad \Rightarrow A \in \psi(U(n))
\end{aligned}
$$

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He probably means

$U(n) \cap GL(n,\mathbb{R}) = O(n,\mathbb{R})$

There seems to be a problem with this interpretation since $\psi(U(n)) \subseteq SO(2n)$ and so can't have

$\Rightarrow A \in GL(2n,\mathbb{R}) \cap \psi(U(n))$

$\Rightarrow O(2n) \subset GL(2n,\mathbb{R}) \cap \psi(U(n))$

$\therefore \boxed{GL(2n,\mathbb{R}) \cap \psi(U(n)) = O(2n)}$

$O(2n) \subseteq \psi(U(n))$