Know the definition of

1. what it means to say X is a left-invariant vector field, P144
2. what it means to say a vector field X is φ-related to a vector field Y, Page 150
3. what it means to say $i: H \to G$ is an immersed Lie Subgroup, now page 157
4. Know how the Lie bracket is defined on T_eG for a Lie group G and be able to say why it makes sense, long winded version page 161
5. Know the definition of the exponential function for an arbitrary Lie group G, $\exp: T_eG \to G$; this should include the last 3 sentences on page 187 + first 8 sentences on page 188.

Know the Proofs of

1. if $X_i \overset{\varphi}{\sim} Y_i$, $i = 1, 2$ then $[X_1, X_2] \overset{\varphi}{\sim} [Y_1, Y_2]$
2. if X, Y are left invariant vector fields so is $[X, Y]$
3. Assuming that for $B \in gl(n)$

$$X^B = \sum_{i,j,k} (x_{ik} B_{kj}) \frac{\partial}{\partial x_{ij}}$$

be able to show that for $B_1, B_2 \in gl(n)$, $X^{[B_1, B_2]} = [X^{B_1}, X^{B_2}]$, $[B_1, B_2] = B_1B_2 - B_2B_1$.

4. Be able to show that if $\varphi: G \to H$ is a Lie group homomorphism then $d_e\varphi: T_eG \to T_eH$ is a Lie algebra homomorphism. Be able to explain why this implies that if $i: H \to G$ is an immersed Lie subgroup then the Lie algebra of ~~H~~ may be identified as a Lie subalgebra of G.
5. Be able to prove the properties of exp summarized in Theorem 11 page 188.

Know how to do the problems you turned in for homework

Def^n/ A vector field X defined on a Lie group G is LEFT INVARIANT iff $\forall x, a \in G$ $d_x l_a (X(x)) = X(ax)$.

Def^n/ Let M and N be manifolds and $\varphi : M \longrightarrow N$ a smooth mapping. If $X \in \Gamma(M)$ and $Y \in \Gamma(N)$ then we say X & Y are φ-related iff for each $x \in X$

$$d_x \varphi (X_x) = Y_{\varphi(x)}$$

That is to say $d\varphi \circ X = Y \circ \varphi$

Def^n/ If H is a Lie group and G is a Lie group and $i : H \longrightarrow G$ is an injective immersion (meaning i and $d_p i$ are injective $\forall p \in H$) and a group homomorphism then we say that H is an immersed Lie subgroup

Def^n/ The set of LIVF on G is denoted $\Gamma_{inv}(G)$, it is a Lie algebra under the bracket on vector fields, $X^v, Y^w \in \Gamma_{inv}(G)$ then $[X^v, X^w]_p(f) \equiv X^v_p (X^w f) - X^w_p (X^v f)$ where $X^v_p \equiv d_e l_p (v)$ for $v \in T_e G$. Now $\exists$ a natural isomorphism of $T_e G$ and $\Gamma_{inv}(G)$, $\Phi(v)(p) \equiv d_e l_p (v) \equiv X^v(p)$ so $\Phi(v) = X^v$ hence $\Phi^{-1}(X^v) = v$. The $T_e G$ inherits a Lie algebra structure under the isomorphism Φ,

$$[v, w] = \Phi^{-1}([\Phi(v), \Phi(w)])$$

It is implicitly defined by $X^{[v,w]} \equiv [X^v, X^w]$.

Def^n/ For each $v \in T_e G$ let $X(p, v) \equiv d_e l_p (v)$ for each $p \in G$. Thus $p \longrightarrow X(p, v)$ is the LIVF such that $X(e, v) = v$. If we denote the sol^n of $p \longmapsto X(p, v)$ by $t \longmapsto \varphi(t, v)$ then

$$\frac{\partial \varphi}{\partial t} = X(\varphi(t, v), v) \qquad \forall t \in \mathbb{R}.$$

Now X is a differential eq^n which depends smoothly on the parameter v and by a standard Th^m in DEq^n's $(t, v) \longrightarrow \varphi(t, v)$ is smooth. We *define* $\exp : T_e G \longrightarrow G$ by

$$\exp(v) \equiv \varphi(1, v)$$

this is smooth by the comments above.

does this say $\varphi(0, v) = e$.

$\varphi(1, v) = \exp(v)$

1.) If $X_i \overset{\varphi}{\sim} Y_i$, $i=1,2$ then $[X_1, X_2] \overset{\varphi}{\sim} [Y_1, Y_2]$.

Let $X_i \in \Gamma(M)$ and $Y_i \in \Gamma(N)$ $i=1,2$ and $\varphi: M \to N$ smooth, we have $d\varphi(X_i) = Y_i \circ \varphi$ for $i=1,2$. We seek to show $d\varphi([X_1, X_2]) = [Y_1, Y_2]\circ\varphi$. Let $g \in C^\infty_p N$

$$
\begin{aligned}
d_p\varphi\left([X_1, X_2]_p\right)(g) &= [X_1, X_2]_p (g\circ\varphi) \\
&= (X_1)_p\left(X_2(g\circ\varphi)\right) - (X_2)_p\left(X_1(g\circ\varphi)\right) \\
&= (X_1)_p\left(Y_2(g)\circ\varphi\right) - (X_2)_p\left(Y_1(g)\circ\varphi\right) \\
&= (Y_1)_{\varphi(p)}\left(Y_2(g)\right) - (Y_2)_{\varphi(p)}\left(Y_1(g)\right) \quad \text{by (II)} \\
&= [Y_1, Y_2]_{\varphi(p)}(g) \quad \Leftrightarrow \quad [X_1, X_2] \overset{\varphi}{\sim} [Y_1, Y_2]
\end{aligned}
$$

$$\therefore \; d\varphi([X_1, X_2]) = [Y_1, Y_2]\circ\varphi$$

Remark: this proof relies on the following identifications,

$$
\begin{aligned}
X \overset{\varphi}{\sim} Y \quad &\Leftrightarrow \quad d_x\varphi(X) = Y_{\varphi(x)} \\
&\Leftrightarrow \quad Y_{\varphi(x)}(g) = d_x\varphi(X)(g) = X_x(g\circ\varphi) \\
&\Leftrightarrow \quad Y(g)(\varphi(x)) = X(g\circ\varphi)(x) \\
&\Leftrightarrow \quad Y(g)\circ\varphi = X(g\circ\varphi)
\end{aligned}
$$

These seem to confuse wether $X(P)$ means the fnct obtained from X acting on fnct P as opposed to the vect. field evaluated at a point P to give tangent vector.

2.) If $\Sigma, \Upsilon \in \Gamma_{INV}(G)$ then $[\Sigma, \Upsilon] \in \Gamma_{INV}(G)$.

<u>Proof</u>: Suppose $\Sigma, \Upsilon \in \Gamma_{INV}(G)$ then $\exists\, v, w \in T_eG$ such that $d_x l_a(\Sigma_x) = \Sigma_{l_a(x)}$ and $d_x l_a(\Upsilon_x) = \Upsilon_{l_a(x)}$ identify $\varphi = l_a$ in part 1.) of test review. Thus

$$\begin{array}{ccc} \Sigma & \overset{l_a}{\sim} & \Sigma \\ \Upsilon & \overset{l_a}{\sim} & \Upsilon \end{array} \Rightarrow [\Sigma, \Upsilon] \overset{l_a}{\sim} [\Sigma, \Upsilon]$$

$$\Rightarrow d_x l_a([\Sigma, \Upsilon]_x) = [\Sigma, \Upsilon]_{l_a(x)}$$

$$\Rightarrow [\Sigma, \Upsilon] \in \Gamma_{INV}(G).$$

<u>Remark</u>: I'd like to find a more direct proof.

3.) Let $B \in gl(n)$ then we may assume (for test) $\Sigma^B = \sum_{ijk} x_{ik} B_{kj} \frac{\partial}{\partial x_{ij}}$

$$[\Sigma^A, \Sigma^B]_c(g) = \Sigma^A_c(\Sigma^B \circ g) - \Sigma^B_c(\Sigma^A \circ g)$$

$$= C_{ik} A_{kj} \frac{\partial}{\partial x_{ij}}\left[x_{lm} B_{mn} \frac{\partial g}{\partial x_{ln}}\right]\Big|_{x=c} - C_{ik} B_{kj} \frac{\partial}{\partial x_{ij}}\left[x_{lm} A_{mn} \frac{\partial g}{\partial x_{ln}}\right]\Big|_{x=c}$$

$$= C_{ik} A_{kj} \delta_{il} \delta_{jm} B_{mn} \frac{\partial g}{\partial x_{ln}} - C_{ik} B_{kj} \delta_{il} \delta_{jm} A_{mn} \frac{\partial g}{\partial x_{ln}} + \cancel{\left(C_{ik} A_{kj} x_{lm} B_{mn} \frac{\partial^2 g}{\partial x_{ij} \partial x_{ln}} - C_{ik} B_{kj} x_{lm} A_{mn} \frac{\partial^2 g}{\partial x_{ij} \partial x_{ln}}\right)}$$

$$= \left(C_{lk} A_{km} B_{mn} - C_{lk} B_{km} A_{mn}\right) \frac{\partial}{\partial x_{ln}}(g)\Big|_{x=c}$$

$$= \left(x_{lk} [A, B]_{kn} \frac{\partial}{\partial x_{ln}}\right)_c g$$

$$= \Sigma^{[A,B]}_c(g).$$

In notes of 9/18/06.1 we do this argument at $c = I$, but I think its not necessary

$$C_{ik} A_{kj} C_{lm} B_{mn} \frac{\partial^2 g}{\partial x_{ij} \partial x_{ln}} \overset{i \leftrightarrow l,\ j \leftrightarrow n}{=} C_{lk} A_{kn} C_{im} B_{mj} \frac{\partial^2 g}{\partial x_{ln} \partial x_{ij}}$$

$$= C_{lk} A_{kn} C_{im} B_{mj}\, \partial^2 g / \partial x_{ij} \partial x_{ln}$$

$$= \boxed{C_{im} B_{mj}} \boxed{C_{lk} A_{kn}}\, \partial^2 g / \partial x_{ij} \partial x_{ln}$$

4a) Be able to show if $\varphi : G \to H$ is a Lie group homomorphism then $d_e\varphi : T_eG \to T_eH$ is a Lie algebra homomorphism.

<u>Proof</u>: $(\varphi \circ l_a)(x) = \varphi(l_a(x)) = \varphi(ax) = \varphi(a)\varphi(x)$.

$$\begin{aligned} d_x\varphi\left(X_G^V(x)\right) &= d_x\varphi\left(d_e l_x(V)\right) \\ &= d_e(\varphi \circ l_x)(V) \\ &= d_e(l_{\varphi(x)} \circ \varphi)(V) \\ &= d_e l_{\varphi(x)}(d_e\varphi(V)) \\ &= X_H^{d_e\varphi(V)}(\varphi(x)) \qquad \therefore X_G^V \overset{\varphi}{\sim} X^{d_e\varphi(V)} \end{aligned}$$

Now we may use our earlier Lemma $[X_1, X_2] \overset{\varphi}{\sim} [Y_1, Y_2]$

$$[X_G^V, X_G^W] \overset{\varphi}{\sim} [X_H^{d_e\varphi(V)}, X_H^{d_e\varphi(W)}]$$

$$\therefore \quad X_G^{[V,W]_G} \overset{\varphi}{\sim} X_H^{[d_e\varphi(V), d_e\varphi(W)]_H}$$

$$\therefore \quad d_x\varphi\left(X_G^{[V,W]}(x)\right) = X^{[d_e\varphi(V), d_e\varphi(W)]_H}(\varphi(x))$$

evaluate $x = e_G$, $\varphi(e) = e_H$ $\quad \therefore \quad d_e\varphi([V,W]_G) = [d_e\varphi(V), d_e\varphi(W)]_H$

Also $d_e\varphi : T_eG \to T_eH$ is linear, thus $d_e\varphi$ is a linear bracket preserving map. It is a Lie algebra homomorphism.

4b) Be able to show if $i : H \to G$ is an immersed Lie subgroup then T_eH may be identified as a subalgebra of T_eG. By definition $i : H \to G$ is a <u>smooth</u> homomorphism $\therefore$ by 4a) $d_e i : T_eH \to T_eG$ is a Lie algebra homomorphism. Thus by 4a,

$$d_e i([V,W]_H) = [d_e i(V), d_e i(W)]_G$$

$$\therefore \; d_e i(T_eH) \subset T_eG$$

And $d_e i$ is injective $\therefore$ $d_e i : T_eH \to i(T_eH)$ is an isomorphism.

$$T_eH \cong d_e i(T_eH)$$

5.) Be able to prove properties of exp given in Th^{m} (11) of pg. 188,

Th^{m} (11) Let G be a Lie group

(1.) If φ is a C^1 one-parameter group in G then φ satisfies

$$\frac{d\varphi}{dt} = \mathbb{X}(\varphi(t), v)$$

where $v = (d\varphi/dt)(0)$.

(2.) If φ is a one-parameter group in G then $\varphi(t) = \exp(tv)$ $\forall t \in \mathbb{R}$ where $v = (d\varphi/dt)(0)$

(3.) If $a \in G$ then the integral curve of the vectorfield $x \longmapsto \mathbb{X}(x, v)$ which passes through a is just

$$t \longmapsto l_a(\exp(tv))$$

(4.) If $v \in T_eG$ then $\exp((t+s)v) = \exp(tv)\exp(sv)$ and $\exp(-tv) = (\exp(tv))^{-1}$ $\forall s, t \in \mathbb{R}$.

(5.) If $T_0(T_eG) \cong T_eG$ then $d_0(\exp)$ is the identity mapping from $T_0(T_eG)$ onto T_eG. Thus $(\exp)$ is locally invertible near $0 \in T_eG$.

Proof:

(1) Consider $\varphi : \mathbb{R} \longrightarrow G$ a C^1, 1-parameter group. Fix any s and take derivative w.r.t. t of $\varphi(t+s)$

$$\begin{aligned}\varphi'(t+s) = \frac{d}{dt}\Big(\varphi(t+s)\Big) &= \frac{d}{dt}\Big(\varphi(s+t)\Big) \\ &= \frac{d}{dt}\Big(\varphi(s)\,\varphi(t)\Big) \\ &= \frac{d}{dt}\Big(l_{\varphi(s)}(\varphi(t))\Big) \\ &= d_{\varphi(t)}\, l_{\varphi(s)}(\varphi'(t))\end{aligned}$$

Take $t=0$ to obtain $\varphi'(s) = d_e\, l_{\varphi(s)}(v)$ where $\varphi'(0) = v$.
thus $\varphi'(s) = \mathbb{X}^v(\varphi(s)) = \mathbb{X}(\varphi(s), v)$ in the other notation.

5.) Continued,

Proof (2.): Let $\varphi : \mathbb{R} \longrightarrow G$ be a one-parameter group in G with $\varphi'(0) = V$. We show $\varphi(s) = \exp(sV)$

$$\exp(sV) \equiv \Theta(1, sV)$$

where Θ satisfies,

$$\frac{d\Theta}{dt}(t) = \Xi(\Theta(t, sV), sV) = \Xi^{sV}(\Theta(t))$$

and $\Theta(0, sV) = e$ (is $\Xi(x) = d_e l_x(sV)$? I think so)

Consider, (s is fixed but arbitrary)

$$\begin{aligned} \frac{d}{dt}\left[\varphi(st)\right] &= s\varphi'(st) \\ &= s\, \Xi^{V}(\varphi(st)) \quad \text{by (1.) of Th}^{m}\text{(11).} \\ &= s\, d_{\varphi(st)} l\,(V) \\ &= d_{\varphi(st)} l\,(sV) \\ &= \Xi^{sV}(\varphi(st)) \end{aligned}$$

Notice $\varphi(s(0)) = \varphi(0) = e$ so this also solves $\frac{d\Theta}{dt} = \Xi^{sV}(\Theta(t))$ hence by uniqueness Th^{m} of DEq^{n}s.

$$\Theta(t) = \varphi(st) = \Theta(t, sV)$$

$$\exp(sV) = \Theta(1, sV) = \varphi(s)$$

$$\therefore \underline{\varphi(s) = \exp(sV)}$$

5.) continued

<u>Proof (3.)</u>: The integral curve through $a \in G$ of the vector field $x \longmapsto \Xi(x, v)$ is $t \longmapsto l_a(\exp(tv))$.

Let $\psi(t) = l_a(\exp(tv))$ then we seek to show that ψ solves Ξ that is $\psi'(t) = \Xi^v(\psi(t))$, $\psi(0) = a$.

$$\psi'(t) = \frac{d}{dt}\left[l_a(\exp(tv))\right]$$

$$= d_{\exp(tv)} l_a\left(\frac{d}{dt}\exp(tv)\right)$$

$$= d_{\exp(tv)} l_a\left(\Xi^v(\exp(tv))\right)$$

since $\Xi^v(\varphi(t)) = \varphi'(t)$ & $\varphi(t) = \exp(tv)$ by (2.).

$$= d_{\exp(tv)} l_a\left(d_e l_{\exp(tv)}(v)\right)$$

$$= d_e\left(l_a \circ l_{\exp(tv)}\right)(v)$$

$$= d_e l_{a\exp(tv)}(v)$$

$$= \Xi^v(a\exp(tv))$$

$$= \Xi^v(\psi(t)) \quad (\therefore \psi \text{ is integral curve of } \Xi^v)$$

and notice $\psi(0) = a\underline{\exp(0)} = a$

how do we show this

5.) continued

Proof: (4) If φ is integral curve of X^v s.t. $\varphi(0) = e$ and

$$\varphi'(t) = X^v(\varphi(t))$$

and φ is a one-parameter group then by (2.)

$$\varphi(t) = \exp(tv)$$

Consider,

$$\begin{aligned} \exp((t+s)v) &= \varphi(t+s) \\ &= \varphi(t)\varphi(s) \\ &= \exp(tv)\exp(sv) \end{aligned}$$

Likewise

$$\begin{aligned} \exp((t-t)v) &= \varphi(t-t) \\ &= \varphi(t)\varphi(-t) \\ &= \exp(tv)\exp(-tv) \\ &= \varphi(0) = e \quad \Rightarrow \underline{\exp(-tv) = [\exp(tv)]^{-1}} \end{aligned}$$

5.) Continued

Proof (5.) : If $T_0(T_eG) \equiv T_eG$ then $d_0(\exp)$ is the identity mapping from $T_0(T_eG)$ onto T_eG. Thus $\exists$ open set $U \subseteq T_eG$ such that $0 \in U$ and $\exp(U)$ is open in G & $\exp: U \longrightarrow \exp(U)$ is a diffeomorphism.

Consider,

$$\frac{d}{dt}\left[\exp(tv)\right] = d_{tv}(\exp)\left(\frac{d}{dt}(tv)\right)$$
$$= d_{tv}(\exp)(v)$$

$$\varphi(t) = \exp(tv)$$
$$\varphi'(t) = X^v(\varphi(t)) \quad \text{by (2.)}$$
$$\varphi'(0) = X^v(\varphi(0)) = X^v(e) = v$$

$$\therefore \left.\frac{d}{dt}\left[\exp(tv)\right]\right|_{t=0} = d_0(\exp)(v) = \varphi'(0) = v$$
$$\therefore d_0(\exp)(v) = v$$
$$\therefore d_0(\exp) = id_{T_eG} = id_{T_0(T_eG)}.$$

The rest of (5.) follows by inverse-fnct. Th$^{\underline{m}}$ for manifold theory.