We consider mappings

$$\ell : G \longrightarrow \mathrm{Aut}(V) \quad \text{where } \ell_g : v \longmapsto \rho(g,v), \; \ell_g(x) = g \cdot x.$$

$$\rho : G \times V \longrightarrow V \quad \text{where} \quad g \cdot v = \rho(g,v)$$

Here V is said to be a G-module if it has a representation on it, We require ρ to be continuous, ℓ_g is the translation by g under the assumed group action. By definition ρ being an action implies

(i). $\rho(e,v) = v$	(ii.) $\rho(g,\rho(h,v)) = \rho(gh,v)$	: ρ - notation
(i.) $ev = v$	(ii) $(gh)v = g(hv)$	: juxtaposition notation Sometimes drop "$\cdot$" in $g \cdot v$.
(i.) $\ell_e = id_v$	(ii) $\ell_g \circ \ell_h = \ell_{gh}$	

Def^n/ A matrix representation of G is a continuous homomorphism $\ell : G \longrightarrow Gl(n, \mathbb{C})$.

Remark: upon a choice of basis for V $\exists$ an obvious identification of $\mathrm{Aut}_{\mathbb{C}}(V)$ with $Gl(n,\mathbb{C})$; $L \longrightarrow [L]$.

Def^n/ A morphism $f : V \longrightarrow W$ between representations is a linear map which is equivariant,

$$f(gv) = g f(v)$$

$\forall g \in G$ and $v \in V$. The set of all such morphisms of G-modules is denoted $\mathrm{Hom}_G(V,W)$.

Remark: G-morphism might be better.

Def^n/ If $U \subseteq V$ is a subspace of V then it is said to be a submodule iff $g \cdot u \in U$ $\forall g \in G$ and $u \in U$

$$\ell_g^U \equiv \ell_g \big|_U \quad \text{(sub representation)}$$

a nonzero rep V is irreducible iff V has no submodule except $\{0\}$ and V.

Def^n/ If V is a complex G-module, an Hermitian inner product $V \times V \longrightarrow \mathbb{C}$, $(u,v) \longmapsto \langle u, v\rangle$ is called G-invariant if $\langle gu, gv\rangle = \langle u, v\rangle$ $\forall g \in G$, $u, v \in V$. A rep. together with a G-invariant inner product is called a unitary representation.

__Remark:__ If we choose an $\langle , \rangle$-orthonormal basis for V a unitary rep. then the associated matrix rep. is a homo $G \to U(n)$.

- missing 11/15/06 notes! (defined Hom_G and did Th^m 1.7, Prop. 1.9)

Th^m(1.7) If V is a rep of compact Lie group G then V has a G-inv innerproduct

__Proof:__ Let $b : V \times V \to \mathbb{C}$ be an inner product on finite dim'l V, we know from linear algebra such a product exists. We use the invariant integral to construct a G-inv. inner product C

$$C(u,v) \equiv \int_G b(gu, gv)\, dg$$

linearity and conjugate linearity follows from the same for b plus the linearity of the invariant integral. Furthermore the integral is monotone $\Rightarrow C(u,v) \geq 0$ $\forall u, v \in V$. Check G-inv., let $h \in G$

$$\begin{aligned}
c(hu, hv) &= \int_G b(hgu, hgv)\, dg \\
&\equiv \int_G f(hg)\, dg \qquad f(hg) = b(hgu, hgv) \\
&= \int_G (f \circ l_h) \\
&= \int_G f(g)\, dg \\
&= \int b(gu, gv)\, dg \\
&= c(u,v).
\end{aligned}$$

$\therefore$ $C : V \times V \to \mathbb{C}$ is a G-invariant innerproduct. //

Proposition (1.9) Let G be a compact Lie group. If V is a submodule of the G-module U, then $\exists$ a complementary submodule W such that $U = V \oplus W$. Each G-module is a direct sum of irred. submodules.

Proof: By Th^m (1.7) $\exists$ $\langle , \rangle : V \times V \to \mathbb{C}$ with $\langle gu, gv \rangle = \langle u, v \rangle$, that is a G-invariant inner product $\langle , \rangle$. Define

$$W = V^\perp \equiv \{ w \in U \mid \langle w, v \rangle = 0 \quad \forall v \in V \}$$

we can show $V^\perp$ is a subspace. Lets verify that W is a G-module hence a submodule of U, let $w \in W$ and $g \in G$

$$\begin{aligned} \langle g \cdot w, v \rangle &= \langle g^{-1} \cdot gw, g^{-1} v \rangle \\ &= \langle w, g^{-1} v \rangle \quad \text{but } g^{-1} v \in V \text{ since } G \cdot V = V. \\ &= 0 \end{aligned}$$

therefore $g \cdot w \in W$ hence $g \cdot W \subset W$.

Now we argue each G-module is a direct sum of irred. submodules. Let $n = \dim(U)$. If $n = 1$ then $U = V$. assume that every G-module of $\dim < n$ is a direct sum of irred reps. Suppose $\dim(U) = n$. If U is irred. then we're done. If U is reducible then $\exists V \subset U$ with $G \cdot V \subset V$ so $U = V \oplus V^\perp$ and by assumption $\dim V \geq 1$ hence as $n = \dim(U) = \dim(V) + \dim(V^\perp)$ it follows

$$\dim(V^\perp) = n - \dim(V) \leq n - 1$$

$\therefore$ $V^\perp$ is irred. $\Rightarrow$ $U = V \oplus V^\perp$ is irred. direct sum decomposition.

Th^{m} (1.10) Let G be any group and V & W irred. modules. Then
(i.) a morphism $f: V \longrightarrow W$ is zero or an isomorphism.
(ii.) Every morphism $f: V \longrightarrow V$ has the form $f = \lambda\, id_V$ for some $\lambda \in \mathbb{C}$.
(iii.) If $V \cong W$ then $\dim_{\mathbb{C}}(\text{Hom}_G(V, W)) = 1$. If $V \not\cong W$ $\dim(\text{Hom}_G(V,W)) = 0$.

Proof: (i) Let $f: V \longrightarrow W$ be a linear map with $f(gv) = gf(v)$ $\forall g \in G$ and $v \in V$. Observe that $\ker(f)$ is a submodule of V. Clearly its a subspace. Then G-inv. follows quickly. Let $v \in \ker(f)$ and $g \in G$,

$$f(gv) = gf(v) = 0 \quad \therefore \quad gv \in \ker f.$$

But since V is irred $\Rightarrow$ $\ker f = \{0\}$ or $\ker f = V$ if $\ker f = V$ then $f = 0$. If $\ker f = \{0\}$ then f is injective. Next observe $\text{Im} f$ is a submodule of W, clearly its a subspace and $f(v) \in \text{Im}(f)$ has,

$$g \cdot f(v) = f(gv) \in \text{Im}(f) \quad \therefore \quad G \cdot \text{Im}(f) \subset \text{Im} f.$$

As W is irred and $f \neq 0 \Rightarrow \text{Im}(f) = W$. $\therefore$ f isomorphism.

(ii) If $f = 0$ then $f(x) = 0 \cdot x = \lambda \cdot Id_V(x)$ $\lambda = 0$.
If $f \neq 0$ then $\exists\, \lambda \neq 0 \in \mathbb{C}$ with $x \neq 0$ such that $f(x) = \lambda x$. Consider then, $W_\lambda = \{y \in V /\ f(y) = \lambda y\}$
Let $y \in W_\lambda$ and $g \in G$

$$f(g \cdot y) = g \cdot f(y) = g \lambda y = \lambda(gy) \quad \therefore \quad gy \in W_\lambda.$$

Hence W_λ is submodule of V which is nontrivial $(x \neq 0)$ thus $W_\lambda = V$ (by irred.) $\therefore$ $f(v) = \lambda v = \lambda\, id_V(v)$ $\forall v \in V$.

(iii) • Suppose $V \cong W$ then $f: V \longrightarrow W \cong V \longrightarrow V$ thus $f = \lambda\, id_V$ $\therefore$ $\text{Hom}_G(V, V) = \text{span}_{\mathbb{C}}\{id_V\}$.

• Suppose $V \not\cong W \Rightarrow f = 0$ by (i) since otherwise $f: V \longrightarrow W$ would be a G-isomorphism giving $V \cong W$ $\rightarrow\leftarrow$

<u>Remark 1.12</u>: If V & W are G-modules then $V \oplus W$ is a G-module w.r.t. $g \cdot (v, w) = (g \cdot v, g \cdot w)$. This is easily seen to provide group action on $V \oplus W$,

$$e \circ (v, w) = (e \cdot v, e \cdot w) = (v, w)$$

$$g \cdot (h \circ (v, w)) = g \circ (hv, hw) = (ghv, ghw) = gh \circ (v, w).$$

linearity also follows, $l_g(v, w) = (l_g v, l_g w)$.

$$l_g^V(v) = A(g)[v]$$

$$l_g^W(w) = B(g)[w]$$

$$l_g^{V \oplus W}(v + w) = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array} \right) \begin{pmatrix} [v] \\ [w] \end{pmatrix}$$

$\boxed{\text{Proposition 1.13} \quad \text{An irred. rep. of an Abelian Lie group is one-dim'l}}$

<u>Proof:</u> If $g \in G$ then $l_g : V \to V$ is a morphism in this case,

$$l_g(hx) = ghx = hgx = h\, l_g(x)$$

But by Schur's Lemma $l_g = \lambda_g \, id_V$. Consider then that <u>any</u> nonzero subspace will be a submodule, $S \triangleleft V$,

$$g \circ S = l_g(S) = \lambda_g \, id_V(S) = \lambda_g S \subseteq S$$

But since V is irred there can only be V as a nonzero submodule $\therefore$ V must be one dim'l. (If it were higher dim'l then take any basis vector form span then that will be submodule in $\rightarrow\leftarrow$ to irred. of V).

<u>Remark</u> $\quad l_{gh}(x) = \lambda_{gh} \, id_V$

$$l_g \circ l_h = \lambda(g)\lambda(h) \, id_V \quad \Rightarrow \lambda(gh) = \lambda(g)\lambda(h)$$

Thus $Hom(G \to \mathbb{C})$ will really $Hom(G, S^1)$

S^1 follows from unitarity.

$\boxed{\text{Remark(1)}\ \ Hom(V,W) \cong V^* \otimes W}$

Define $\Theta : V^* \otimes W \longrightarrow Hom(V,W)$ by

$$\Theta(v^* \otimes w)(u) \equiv v^*(u)\, w$$

Choose a basis $\{v_i\}$ of V and $\{w_j\}$ of W.

Then for $f \in Hom(V,W)$ we calculate

$$f(v_i) = \sum_j \Omega_{ji}\, w_j$$

then consider,

$$\Theta\Big(\sum_{i,k} \Omega_{ik}\, v_k^* \otimes w_i\Big)(u) = \sum_{i,k} \Omega_{ik}\, \Theta(v_k^* \otimes w_i)(u)$$

$$= \sum_{i,k} \Omega_{ik}\, v_k^*(u)\, w_i$$

Then suppose $u = v_\ell$ and thus $v_k^*(v_\ell) = \delta_{k\ell}$ yielding

$$\Theta\Big(\sum_{i,k} \Omega_{ik}\, v_k^* \otimes w_i\Big)(v_\ell) = \sum_i \Omega_{i\ell}\, w_i = f(v_\ell).$$

Thus proving $\boxed{f = \Theta\Big(\sum_{i,k} \Omega_{ik}\, v_k^* \otimes w_i\Big)}$ $(*)$

for any $f \in Hom(V,W)$ this gives the way to write f in terms of its matrix (Ω_{ji}) and the $V^* \otimes W \overset{\Theta}{\cong} Hom(\ ,\ )$.

$\boxed{\text{Remark (2)}\ \ \Theta : V^* \otimes W \longrightarrow Hom_G(V,W) \text{ is a } G\text{-morphism}}$

$$\begin{aligned}
\Theta(g\cdot(v^* \otimes w))(u) &= \Theta((g\cdot v^*) \otimes (g \cdot w))(u) \\
&= (g\cdot v^*)(u)\; g\cdot w \\
&= v^*(g^{-1}u)\; g\cdot w \\
&= g\cdot[\,v^*(g^{-1}u)\, w\,] \\
&= g\cdot \Theta(v^* \otimes w)(g^{-1}u) \\
&= [\,g\cdot \Theta(v^* \otimes w)\,](u).
\end{aligned}$$

Since $G \circ Hom(V,W)$ works by $(g\cdot f)(u) = \underset{W}{\underset{\uparrow}{g\cdot}} f(\underset{V}{\underset{\uparrow}{g^{-1}\cdot}} v)$.

Remark (3): $\mathrm{Hom}(V,V) \cong V^* \otimes V$

Def^n/ The TRACE MAPPING $Tr : V^* \otimes V \longrightarrow \mathbb{C}$ via $Tr(v^* \otimes w) = v^*(w)$

Remark : the usual idea of Trace is hidden here. Choose basis $\{v_i\}$ of V and $\Rightarrow$ dual basis $\{v_k^*\}$ of V^*. But first use Θ to map over to $\mathrm{Hom}(V,V)$. Let $f \in \mathrm{Hom}(V,V)$,

$$\begin{aligned} Tr(f) &= Tr(\Theta^{-1}(f)) \\ &= Tr\Big(\sum_{i,k} r_{ik}\, v_k^* \otimes v_i\Big) \quad \text{using } (*) \text{ on } 6. \\ &= \sum_{i,k} r_{ik}\, v_k^*(v_i) \\ &= \boxed{\sum_i r_{ii} = \mathrm{Trace}(f), \quad f(v_i) = \sum_j r_{ji} v_j} \end{aligned}$$

Def^n/ The fixed point set of a representation V of Lie group G is
$$V^G = \{ v \in V \mid g \cdot v = v \quad \forall g \in G \}$$

Def^n/ Integration of W-valued functions of G. Suppose $f : G \to W$ with $f \in C^0(G, W)$. Then $\exists!$ vector which we denote
$$\int_G f(g)\, dg \in W$$
such that $\forall \alpha \in W^*$
$$\alpha\Big(\int_G f(g)\, dg\Big) = \int_G (\alpha \circ f)(g)\, dg$$
where on the RHS $\alpha \circ f : G \xrightarrow{f} W \xrightarrow{\alpha} \mathbb{C}$ and $\int_G dg$ is the left-invariant integral on G for $\mathbb{C}$-valued fncts.

Moreover, in terms of the corresponding innerproduct on W, $\alpha(x) = \langle x, \cdot \rangle$
$$\alpha\Big(\int_G f(g)dg\Big)(w) = \Big\langle \int_G f(g)dg, w \Big\rangle = \int_G \langle f(g), w\rangle dg$$

Continuing to discuss V-valued fnct. integration over G

We define a projection P onto V^G as follows

$$P(v) = \int_G (g \cdot v)\, dg$$

Consider then $h \in G$,

$$\left.\begin{aligned} h \cdot P(v) &= h \cdot \int_G (g \cdot v)\, dg \\ &= \int h \cdot (g \cdot v)\, dg \\ &= \int (hg) \cdot v\, dg \\ &= \int (hg) \cdot v\, d(hg) \\ &= \int \bar{g} \cdot v\, d\bar{g} \\ &= P(v). \end{aligned}\right\}$$

technically, it seems this is a bit bogus w/o some appeal to running these facts through a $\alpha \in V^*$ to borrow from $\int_G : C^0(G, \mathbb{C}) \to \mathbb{C}$.

<u>Comment</u>: there are two main ways to view V-valued integration,

(i) $\langle \int_G f(g)\,dg, w \rangle = \int_G \langle f(g), w \rangle\, dg$

(ii) trace ? <u>But</u> $\mathrm{Tr} : V^* \otimes V \longrightarrow \mathbb{C}$ (not $V \longrightarrow \mathbb{C}$).

Continuing our discussion of $P(v) = \int_G (g \cdot v)\, dg$. Claim $P(V) \subset V^G$.

Let $v \in V$ and $h \in G$ consider

$$\begin{aligned} \langle h P(v), w \rangle &= \langle h \int (g \cdot v)\, dg, w \rangle \\ &= \langle \int hg \cdot v\, dg, w \rangle \\ &= \langle \int g \cdot v\, dg, h^{-1} w \rangle \\ &= \int \langle g \cdot v, h^{-1} w \rangle\, dg \\ &= \int \langle hg\, v, w \rangle\, dg \\ &= \int \langle gv, w \rangle\, dg \\ &= \langle \int_G g \cdot v\, dg, w \rangle = \langle P(v), w \rangle \quad \forall w. \end{aligned}$$

$\therefore\ hP(v) = P(v).$

Showing $P(v) = \int g\cdot v \, dg$ is a projection

Let $v \in V^G$ so $g\cdot v = v$ $\forall g \in G$. Let $w \in V$,

$$\begin{aligned} \langle P(v), w \rangle &= \left\langle \int_G (g\cdot v)\, dg, w \right\rangle \\ &= \int_G \langle g\cdot v, w \rangle \, dg \\ &= \int_G \langle v, w \rangle \, dg \\ &= \langle v, w \rangle \int_G dg = \langle v, w \rangle \Rightarrow P(v) = v \end{aligned}$$

Thus $P|_{V^G} = id_{V^G}$.

(Remark: V^G is submodule $\Rightarrow V = V^G \oplus (V^G)^\perp$)

$$\boxed{\text{Proposition: } [Hom(V, V)]^G = Hom_G(V, V)}$$

Proof: $Hom(V, V)$ has action as follows $f \in Hom(V, V)$ where

$$(g\circ f)(x) = g\cdot f(g^{-1}x)$$

$$\begin{aligned} f \in [Hom(V, V)]^G &\Leftrightarrow g\cdot f(g^{-1}x) = f(x) \quad \forall x \in V, \forall g \in G \\ &\Leftrightarrow g^{-1} f(x) = f(g^{-1}x) \quad \forall x \in V, \forall g \in G \\ &\Leftrightarrow g f(x) = f(gx) \quad \forall x \in V, \forall g \in G \\ &\Leftrightarrow f \in Hom_G(V, V). \end{aligned}$$

11/29/06.3

$\boxed{\begin{array}{l}\text{Th}^{m}\ (4.2)\ \text{Let } V \text{ be an irrep. of a compact Lie group } G \\ \text{then } \forall f \in \text{Hom}(V,V) \\ \displaystyle\int_G (g\cdot f)\,dg = \frac{1}{\dim V}\,\text{Trace}(f)\,\text{id}_V\end{array}}$

Proof: For $f \in \text{Hom}(V,V)$ note $P(f) = \int_G (g\cdot f)dg \in \text{Hom}_G(V,V)$ thus $P(f) : V \longrightarrow V$ is a morphism & by Schur's Lemma

$$P(f) = \lambda_f \, \text{id}_V \quad \text{for some } \lambda_f \in \mathbb{C}.$$

observe that,

$$\begin{aligned}
(\dim_{\mathbb{C}} V)\,\lambda_f &= \text{Tr}(\lambda_f \, \text{id}_V) \\
&= \text{Tr}(P(f)) \\
&= \text{Tr}\left(\int_G (g\cdot f)dg\right) \qquad g\cdot f \in \text{Hom}(V,V) \\
&= \int_G \text{Tr}(g\cdot f)\,dg \qquad \text{Tr}(g\cdot f) \in \mathbb{C}. \\
&= \int_G \text{Tr}(g\circ f\circ l_{g^{-1}})\,dg \\
&= \int_G \text{Tr}(l_g\circ f\circ l_{g^{-1}})\,dg \\
&= \int_G \text{Tr}(f)\,dg \\
&= \text{Tr}(f)\int_G dg = \text{Tr}(f) \quad \therefore \boxed{\lambda_f = \frac{\text{Tr}(f)}{\dim_{\mathbb{C}} V}}
\end{aligned}$$

$$\therefore \quad P(f) = \boxed{\int_G (g\cdot f)dg = \frac{\text{Tr}(f)}{\dim_{\mathbb{C}} V}\,\text{id}_V}$$

Representative Functions

Notice if a representation space V has basis $\{V_i\}$ of a Lie group G then we have a corresp. matrix rep.

$$g \longrightarrow (\Omega_{ij}(g))$$

from G into $Gl(n, \mathbb{C})$ defined by

$$g \circ V_j = \sum_i \Omega_{ij}(g) V_i$$

Then it follows,

$$\boxed{\Omega_{ij}(g) = V_i^*(g \circ V_j)}$$

> Def^n/ A function $f : G \longrightarrow \mathbb{C}$ relative to a representation space V of a Lie group G into $\mathbb{C}$ is called a <u>representative function</u> iff $\exists \varphi \in V^*, v \in V$ such that
>
> $$f(g) = \varphi(g \circ v) \quad \forall g \in G.$$

<u>Remark</u>: defining $\Omega_{ij} : G \longrightarrow \mathbb{C}$ by $g \circ V_j = \sum_i \Omega_{ij}(g) V_i$ we again notice,

$$\Omega_{ij}(g) = V_i^*(g \circ V_j)$$

So here $\varphi = V_i^*$ and $V = V_j$ in the Def^n above. This shows that Ω_{ij} is a representative fnct.

- Need to compare this discussion with p. 124-125. (this may be "baby version?")

<u>Proposition 4.4</u>: Let G be rep. by V and take $\varphi \in V^*$, $v \in V$, $f \in Hom(V,V)$

$$\int_G \varphi(g(f(g^{-1}v)))\,dg = \frac{Tr(f)}{\dim V}\varphi(v)$$

<u>Proof</u>: Use (4.2) $\int (g\cdot f)dg = \frac{Tr(f)}{\dim V} id_V$. Evaluate on v

$$\int (g\cdot f)(v)\,dg = \frac{Tr(f)}{\dim V} v$$

Eval. 4.2 at v.

(DR. FULP's Proof Elaborates on this point.)

Act by φ,

$$\varphi\left(\int (g\cdot f)(v)\,dg\right) = \varphi\left(\frac{Tr(f)}{\dim V} v\right)$$

linearity of $\varphi \in V^*$

$$\int \varphi((g\cdot f)(v))\,dg = \frac{Tr(f)}{\dim V} v$$

$(g\cdot f)(x) \equiv g\cdot f(g^{-1}x)$.

$$\boxed{\int \varphi(g\cdot f(g^{-1}v))\,dg = \frac{Tr(f)}{\dim V} v}$$

//.

<u>DR. FULP's PROOF</u>:

$$\beta : Hom(V,V) \longrightarrow V \qquad \beta(f) \equiv f(v), \quad \beta = eval_v.$$

$$\beta\left(\int (g\cdot f)\,dg\right) = \int \beta(g\cdot f)\,dg$$

Now act by α,

$$\alpha\left[\beta\left(\int (g\cdot f)dg\right)\right] = (\alpha\circ\beta)\left(\int_G (g\cdot f)\,dg\right) = \int_G (\alpha\circ\beta)(g\cdot f)\,dg.$$

$$= \alpha\left(\int_G \beta(g\cdot f)dg\right) \quad \forall \alpha \in V^*$$

This means that

$$\left(\int (g\cdot f)\,dg\right)(v) = \int_G (g\cdot f)(v)\,dg.$$

We just verified it by comparing components via dual vector evaluation.

$\boxed{\text{Th}^{m}\,(4.5)\ \text{Let } V \text{ be an irrep of } G.\ \text{For } v,w,\alpha,\beta \in V,}$

(i.) $$\int_G \langle g f(g^{-1}v), w\rangle \, dg = \frac{Tr(f)}{\dim V}\langle v,w\rangle$$

(ii) $$\int_G \langle g^{-1}v, \alpha\rangle\langle g\beta, w\rangle \, dg = \frac{1}{\dim V}\langle \beta,\alpha\rangle\langle v,w\rangle$$

<u>Proof</u>: We know $\int_G \varphi(g(f(g^{-1}v)))dg = \frac{Tr(f)}{\dim V}\varphi(v)$.

It appears $\varphi(v) = \langle v, w\rangle$ may work

$$\int \varphi(g(f(g^{-1}v)))\,dg = \int \langle g(f(g^{-1}v)), w\rangle\, dg$$

$$= \frac{Tr(f)}{\dim V}\varphi(v) = \frac{Tr(f)}{\dim V}\langle v,w\rangle \quad // \text{ on (i.)}$$

Next consider $f(x) = \langle x, \alpha\rangle \beta$ (trick.)

$$\int \langle g(f(g^{-1}v)), w\rangle dg = \int \langle g\cdot\langle g^{-1}v, \alpha\rangle\beta, w\rangle\, dg$$

$$= \int \langle \langle g^{-1}v,\alpha\rangle\beta, g^{-1}w\rangle\, dg$$

$$= \int \langle g^{-1}v,\alpha\rangle\langle \beta, g^{-1}w\rangle\, dg$$

$$= \int \langle g^{-1}v,\alpha\rangle\langle g\beta, w\rangle\, dg \quad \underline{LHS\ (ii)}$$

$$= \frac{Tr(f)}{\dim V}\langle v,w\rangle \quad \text{need } Tr(f) = \langle \beta,\alpha\rangle.$$

<u>Choose $\langle , \rangle$ O.N. basis $\{v_i\}$</u>

$$Tr(f) = \sum_i f(v_i)\cdot v_i$$

$$= \sum_i \langle v_i, \alpha\rangle \beta \cdot v_i$$

$$= \sum_i \bar{\alpha}_k \underbrace{\langle v_i, v_k\rangle}_{\delta_{ik}} \beta_i$$

$$= \sum_i \beta_k \bar{\alpha}_k$$

$$= \langle \beta, \alpha\rangle \checkmark$$

$\boxed{\text{Th}^m\ (4.6)}$ Let $V \not\cong W$ irreps of G with G-invariant inner products $\langle\, ,\, \rangle_V$ and $\langle\, ,\, \rangle_W$ on $V + W$,

$$\int_G \overline{\langle g\alpha, v\rangle_V}\, \langle g\beta, w\rangle_W \, dg = 0$$

$\forall\, \alpha, v \in V$ and $\beta, w \in W$.

<u>Proof</u>: We define a mapping $b(u,v)$ in hope of showing $b = 0$

$$b(v,w) \equiv \int_G \overline{\langle g\alpha, v\rangle_V}\, \langle g\beta, w\rangle_W \, dg$$

This ~~b is an inner-product~~, notice (No it's a weird $V \times \overline{W}^* \longrightarrow \mathbb{C}$ map.)

$$b(cv, w) = c\, b(v,w) \quad \& \quad b(v, cw) = \bar{c}\, b(v,w).$$

I believe this is $b : (V+W) \times (V+W) \longrightarrow \mathbb{C}$. Notice $h \in G$,

$$\begin{aligned} b(hv, hw) &= \int_G \overline{\langle g\alpha, hv\rangle} \langle g\beta, hw\rangle\, dg \\ &= \int_G \overline{\langle h^{-1}g\alpha, v\rangle} \langle h^{-1}g\beta, w\rangle\, dg \\ &= \int_G \overline{\langle g\alpha, v\rangle} \langle g\beta, w\rangle\, dg \\ &= b(v,w). \end{aligned}$$

left-invariance.

Restrict b to $V \times \overline{W}$ and define $\hat{b} : V \longrightarrow \overline{W}^*$ by

$$\hat{b}(v) \equiv b(v, \cdot) \quad \text{aka} \quad \hat{b}(v)(w) \equiv b(v,w).$$

now we show $\hat{b}$ is a morphism,

$$\hat{b}(hv)(w) = b(hv, w) = b(v, h^{-1}w) = \hat{b}(v)(h^{-1}w) = (h \cdot \hat{b})(w).$$

Hence $\hat{b}(hv) = h \cdot \hat{b}(v)$. $\therefore\ \hat{b}$ is a morphism from V to $\overline{W}^* \cong W$ (it can be shown) thus $\hat{b} = 0$ since $\hat{b} : V \longrightarrow \overline{W}^* \cong W \not\cong V$ and since V, W irreps there are either isomorphisms or zero, $\hat{b}$ must be zero.

Then $\hat{b} = 0 \Rightarrow \hat{b}(v)(w) = b(v,w) = 0 \quad \forall\, v, w$

Thus the Th^m follows.

Remark: Showing $\overline{W}^* \cong W$ would be nice.

Formulas 4.8 on matrix representations

Define inner product on $C^0(G, \mathbb{C})$ by (Side Comment)

$$\langle \varphi, \psi \rangle = \int_G \varphi(g)\overline{\psi(g)}\,dg$$

Then let $\{v_1, v_2, \dots, v_m\}$ and $\{w_1, w_2, \dots, w_n\}$ be orthonormal bases of V and W which are irreps of G (relative to $\langle , \rangle_V$ & $\langle , \rangle_W$) Define the matrix reps from V,

$$g \circ v_j = \sum_i \Omega_{ij}^V(g)\, v_i$$

$$g \circ w_j = \sum_i \Omega_{ij}^W(g)\, w_i$$

That is to say

$$\boxed{\Omega_{ij}^V(g) = \langle g \circ v_j, v_i \rangle_V \quad \& \quad \Omega_{ij}^W(g) = \langle g \cdot w_j, w_i \rangle}$$

The formulas 4.8 follow,

$$\int \overline{\Omega_{kl}^V(g)}\, \Omega_{ij}^V(g)\,dg = \int_G \overline{\langle g \cdot v_l, v_k \rangle} \langle g \circ v_j, v_i \rangle\, dg$$

$$= \int_G \langle v_k, g v_l \rangle \langle g v_j, v_i \rangle\, dg$$

$$= \int_G \langle g^{-1} v_k, v_l \rangle \langle g v_j, v_i \rangle\, dg$$

Th^{m} (4.5)

$$= \frac{1}{\dim V} \langle v_l, v_j \rangle \langle v_k, v_i \rangle$$

$$= \frac{1}{\dim V} \delta_{lj}\, \delta_{ki}$$

Next for $V \not\cong W$

$$\int \overline{\Omega_{kl}^W(g)}\, \Omega_{ij}^V(g)\,dg = \int \overline{\langle g \cdot w_l, w_k \rangle} \langle g \cdot v_j, v_i \rangle$$

$$= 0 \quad \text{by Th}^{m}\ (4.6)$$

$V \longrightarrow W$ same

$W \longrightarrow V$ story though.

Characters of compact Lie group

Def^n/ Let $g \longrightarrow l_g$ be a rep. of a compact Lie group G on $\mathrm{Aut}(V)$. Then we define the character of the rep. is the fnct $\chi_V : G \longrightarrow \mathbb{C}$ defined by

$$\chi_V(g) = \mathrm{trace}(l_g)$$

Th^m/ A rep. V of G is determined by its character χ upto isomorphism

$$\mathrm{Irr}(G,\mathbb{C}) = \{\text{all finite dim'l irrep. of } G\} / \text{isomorphisms of vector spaces.}$$

Proof: Recall if V is irred. then $V \cong W$ for some unique $W \in \mathrm{Irr}(G,\mathbb{C})$. A general finite dim'l rep. of G we know

$$V = \bigoplus_j n_j V_j$$

where for each j we have V_j a submodule of V and n_j = multiplicity of V_j & $j \neq k \Rightarrow V_j \not\cong V_k$. This decomp. is not unique due to ordering, however $\exists!$ thing in $\mathrm{Irr}(G,\mathbb{C})$ matching V. Moreover $V_j \cong W_j \in \mathrm{Irr}(G,\mathbb{C})$ and
$$n_j = \dim_{\mathbb{C}}(\mathrm{Hom}_G(V_j, V)) = \dim_{\mathbb{C}}(\mathrm{Hom}_G(W_j, V))$$

$$\therefore \quad V \cong \bigoplus_j n_j W_j$$

Indeed and much more interestingly,

$$\langle \chi_V, \chi_{W_j} \rangle \equiv \int \chi_V(g)\, \overline{\chi_{W_j}(g)}\, dg$$
$$= \sum_k n_k \int \chi_{W_k} \overline{\chi}_{W_j}$$
$$= \sum_k n_k \delta_{kj} = n_j$$