LIE GROUP TEST I

1. (a) What does it mean to say that a vector field X is ϕ related to a vector field Y?
(b) Assume that X_1, X_2 are vector fields on a manifold M and that Y_1, Y_2 are vector fields on a manifold N. Also assume that $\phi : M \to N$ is a smooth mapping and that X_i is ϕ related to Y_i for $i = 1, 2$. Show that $[X_1, X_2]$ is ϕ related to $[Y_1, Y_2]$.
(c) Let X and Y be left-invariant vector fields on a Lie group G. Show that $[X, Y]$ is also left-invariant.

2. (a) Let G be a Lie group. For $v, w \in T_eG$, explain how the bracket $[v, w]$ is defined and discuss what is needed for the definition to be meaningful.
(b) Let $B_1, B_2 \in gl(n, R)$ and write

$$\hat{B}_1 = \sum_{i,j=1}^{n} (B_1)_{ij} \left(\frac{\partial}{\partial x_{ij}}|_e\right) \quad and \quad \hat{B}_2 = \sum_{i,j=1}^{n} (B_2)_{ij} \left(\frac{\partial}{\partial x_{ij}}|_e\right)$$

where (x_{ij}) are the components of the usual global chart on $Gl(n, R)$. Describe how you would compute $[\hat{B}_1, \hat{B}_2]$ as a tangent vector at the identity e of $Gl(n, R)$ **using only the definition in 2.(a).** Note that this is not the matrix $B_1B_2 - B_2B_1$ although, according to one of our theorems, it can be identified with it

3. Assume that ϕ is a one-parameter Lie group in a Lie group G.
(a) Show that $\phi'(t) = X^v(\phi(t))$ for $v = \phi'(0)$ and for all $t \in R$.
(b) Show that $\phi(s) = exp(sv)$ for some $v \in T_eG$ and for all $s \in R$.
(c) Explain how you know that the one-parameter groups fill up a neighborhood of the identity in G, that is, how do you know that there is an open set U of the identity such that every point of U lies on the image of a one-parameter group in G. Can an element of U lie on two one-parameter groups? Uniqueness Th.

THE REST OF THE TEST IS ON THE BACK OF THE PAGE

4. (a) Show that the function $\phi : C \to gl(2, R)$ defined by

$$\phi(z) = \begin{pmatrix} Re(z) & -Im(z) \\ Im(z) & Re(z) \end{pmatrix}$$

satisfies the equations $\phi(zw) = \phi(z)\phi(w)$ and $\phi(\overline{z}) = \phi(z)^t$ for $z, w \in C$.
(b) Let $\psi : gl(2, C) \to gl(4, R)$ be defined by

$$\psi(A) = \begin{pmatrix} \phi(A^1_1) & \phi(A^1_2) \\ \phi(A^2_1) & \phi(A^2_2) \end{pmatrix}.$$

Show that $\psi(A^\dagger) = \psi(A)^t$ and use this to show that $\psi(U(2)) = SO(4, R)$.

THE FOLLOWING PROBLEM MAY BE SUBSTITUTED FOR ANY ONE PART OF ANY ONE OF THE PROBLEMS ABOVE.

Assume that G is a group which is also a manifold and that $\mu : G \times G \to G$ is defined by $\mu(g, h) = gh$ for $g, h \in G$. Show the the inversion mapping ι of G defined by $\iota(g) = g^{-1}, g \in G$, is smooth provided μ is smooth.

fix g $\quad \mu(h, h^{-1}g) = hh^{-1}g = g$