JAMES COOK

1.) a.) Let $\phi : M \to N$ be a smooth mapping then let $X \in \Gamma(M)$ and $Y \in \Gamma(N)$ then we say $X \overset{\varphi}{\sim} Y$ iff $\forall x \in M$,

$$d_x\varphi(X_x) = Y_{\varphi(x)} . \Leftrightarrow X(g \circ \varphi) = Y(g)\circ\varphi$$

b.) Let $g \in C^\infty_{\varphi(x)} N$, and $x \in M$,

$$d_x\varphi([X_1, X_2]_x)(g) = [X_1, X_2]_x(g\circ\varphi)$$

$$= (X_1)_x(X_2(g\circ\varphi)) - (X_2)_x(X_1(g\circ\varphi))$$

$$= (X_1)_x(Y_2(g)\circ\varphi) - (X_2)_x(Y_1(g)\circ\varphi)$$

$$= (Y_1)_{\varphi(x)}(Y_2(g)) - (Y_2)_{\varphi(x)}(Y_1(g))$$

$$= [Y_1, Y_2]_{\varphi(x)}(g) \Rightarrow \underline{[X_1, X_2] \overset{\varphi}{\sim} [Y_1, Y_2]}$$

c.) Let $X, Y \in \Gamma_{INV}(G)$ then $\exists v, w \in T_eG$ s.t. $X_x = d_e l_x(v)$ and $Y_x = d_e l_x(w)$ $\forall x \in G$. which followed from the def$^{\underline{n}}$ that $X_{ax} = d_x l_a(X_x)$

Hence $X \overset{l_a}{\sim} X$ and $Y \overset{l_a}{\sim} Y$ thus

$\xrightarrow{\text{by (b.)}}$ $[X, Y] \overset{l_a}{\sim} [X, Y]$ but this means

$$d_x l_a([X, Y]_x) = [X, Y]_{l_a(x)} = [X, Y]_{ax}$$

$$\therefore \quad [X, Y] \in \Gamma_{INV}(G)$$

2 (a.) To begin we defined the Lie bracket of vector fields on $\Gamma_{INV}(G)$. We noted that due to the f-la $\Xi^V(x) = d_e l_x(V)$ it followed that $\Gamma_{INV}(G)$ was finite-dim'l and moreover obtained the structure of a vect-space. In fact we saw in hwk. that the vect-field bracket (used in 1b). is a Lie bracket satisfying i.) skew ii.) bilinearity iii). Jacobi. Then the f-la $d_e l_x(V) = \Xi^V(x)$ suggests an isomorphism

$$\Phi : T_eG \longrightarrow \Gamma_{INV}(G)$$

$$\Phi(V)(x) = d_e l_x(V). = \Xi^V(x)$$

Which induces a bracket on T_eG as follows,

implicitly: $[\Xi^V, \Xi^W] = \Xi^{[V,W]}$

explicitly: $[V,W] = \Phi^{-1}[\Phi(V), \Phi(W)]$
$= \Phi^{-1}[\Xi^V, \Xi^W]$

(2b) $\hat{B}_1 = \sum_{i,j} (B_1)_{ij} \frac{\partial}{\partial x_{ij}}\Big|_e \qquad \hat{B}_2 = \sum_{i,j} (B_2)_{ij} \frac{\partial}{\partial x_{ij}}\Big|_e$ ③

I suppose we'd take $f \in C^\infty_I(Gl(n))$ and then evaluate, at $x \in Gl(n)$

$$[\hat{B}_1, \hat{B}_2]_x f \overset{\star}{=} (\hat{B}_1)_x (\hat{B}_2 f) - \hat{B}_2(\hat{B}_1 f)$$

Note $\hat{B}_1, \hat{B}_2$ not defined at x!

then after some calculation we'd find

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$$[\hat{B}_1, \hat{B}_2]_x f = \widehat{[B_1, B_2]} f$$

where we'd identify that the coefficients multiplying $\frac{\partial}{\partial x_{ij}}\Big|_e$ make-up $[B_1, B_2]$.

① define LI VF's by $X^{\hat{B}_1}, X^{\hat{B}_2}$

② Bracket these $[X^{\hat{B}_1}, X^{\hat{B}_2}]$ as you do in $\star$

③ Then $[\hat{B}_1, \hat{B}_2] = [X^{\hat{B}_1}, X^{\hat{B}}](e)$

3) Assume ϕ is a one-parameter Lie group in a Lie group G

a.) We know $\phi(s+t) = \phi(s)\,\phi(t)\ \forall s,t$ and also that ϕ smooth.

$$\phi(s+t) = \phi(s)\,\phi(t) = l_{\phi(s)}\,\phi(t)$$

$$\phi'(s+t) = d\,l_{\phi(s)}\left(\phi'(t)\right)$$

$\underline{t=0}$
$$\phi'(s) = d_e\, l_{\phi(s)}\left(\phi'(0)\right)$$
$$= d_e\, l_{\phi(s)}(V)$$
$$= X^V_{\phi(s)}$$

$\therefore$ ϕ solves the LIVF X^V.

b.)

(3b) Show $\phi(s) = \exp(sv)$. We know

$\theta : \mathbb{R} \to G$

$\exists\ \cancel{\theta} : \mathbb{R} \longrightarrow G$ with

definition of $\exp(sv)$ is

$\cancel{\theta}\ \theta(1, sv) = \exp(sv)$ & $\theta'(0) = sv$.

Consider $\boxed{\theta(1, sv) = \exp(sv)}$ & $\theta'(0) = sv$

where $\frac{d\theta}{dt} = X^{sv}(\theta(t))$, $\theta(0) = e$

~~$\phi(t) = \exp(tv)$~~ Claim if $\phi(t)$ is one-param. group with $\phi'(0) = v$

$$\boxed{\theta(t) = \phi(st)}.$$

(−3)

Now $\theta'(0) = sv$. If we can show $\phi'(0) = sv$ then $\theta(t) = \phi(st)\ \forall t$
$\Rightarrow \theta(1) = \phi(s) = \exp(sv)$ the desired result.

<u>Show $\phi'(0) = sv$</u>

$\frac{d}{dt}[\phi(st)]\Big|_{t=0} = s\,\phi'(st)\Big|_{t=0} = s\,\phi'(0) = sv$

$$\frac{d}{dt}\left[\phi(st)\right]\Big|_{t=0} = s\,\cancel{\frac{d}{dt}}\left[\phi(st)\right]\Big|_{t=0}$$

by 3a.

$$= s\ d_e \cancel{\ell}(v)$$

$$= d_e \ell(sv)$$

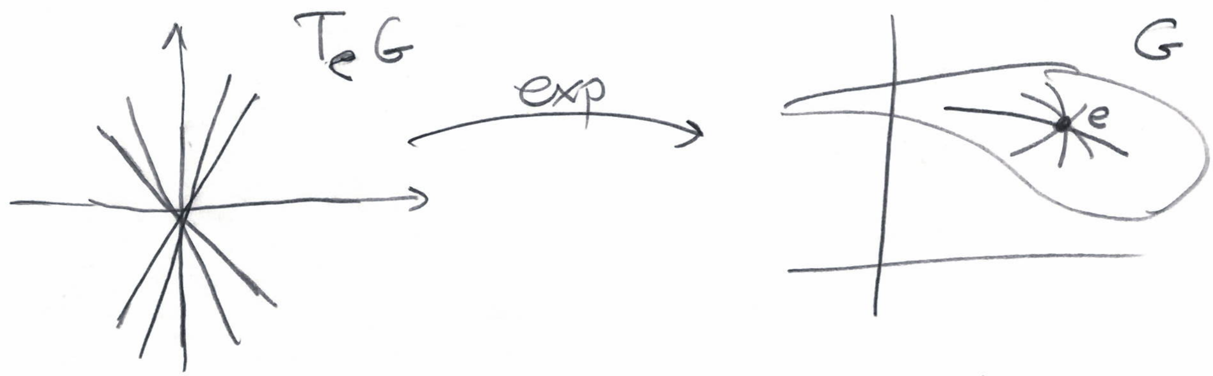
$$= X^{sv} \Rightarrow \phi'(st) = sv.$$

bad notation

if $w = \frac{d}{dt}[\varphi(st)]\Big|_{t=0}$ your equation says $w = sw$ which says $s = 1$.

(3c) We proved that one parameter groups had domains extendable to $\mathbb{R}$ so $\phi : \mathbb{R} \longrightarrow G$. Also we know that $\phi(s+t) = \phi(s)\phi(t)$ and $\phi(0) = e$ since ϕ a homomorphism of $(\mathbb{R}, +)$ to G.

Also from 3b.) we know that $\phi(s) = \exp(sv)$ for $v = \phi'(0)$.



OK We proved that exp was a local diffeomorphism at e thus we have for some $\tilde{U} \overset{\text{open}}{\subset} T_e G$

$$\exp : \tilde{U} \longrightarrow \exp(\tilde{U}) \equiv U$$

open since diffeomorphisms are open maps.

Also exp is injective on $\tilde{U}$.

should say more clearly $w \in U \Rightarrow \exists\, v \in \tilde{U} \ni \exp(v) = w$ $\therefore$ if $\varphi(t) = \exp(tv)$ $\varphi(1) = w \Rightarrow$

cannot have element of U lie on two one-parameter groups.

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4 a) $\varphi : \mathbb{C} \longrightarrow gl(2, \mathbb{R})$

$$\varphi(zw) = \begin{pmatrix} Re(zw) & -Im(zw) \\ Im(zw) & Re(zw) \end{pmatrix}$$

$$\varphi(z)\varphi(w) = \begin{pmatrix} Re\, z & -Im\, z \\ Im\, z & Re\, z \end{pmatrix}\begin{pmatrix} Re\, w & -Im\, w \\ Im\, w & Re\, w \end{pmatrix}$$

$$= \begin{pmatrix} Re\,z\,Re\,w - Im\,z\,Im\,w & -Re\,z\,Im\,w - Im\,z\,Re\,w \\ Im\,z\,Re\,w + Re\,z\,Im\,w & -Im\,z\,Im\,w + Re\,z\,Re\,w \end{pmatrix}$$

$$= \begin{pmatrix} Re(zw) & -Im(zw) \\ Im(zw) & Re(zw) \end{pmatrix}$$

$$= \varphi(zw).$$

Since $zw = (Re\,z + i\,Im\,z)(Re\,w + i\,Im\,w)$

$$= (\underbrace{Re\,z\,Re\,w - Im\,z\,Im\,w}_{Re(zw).}) + i(\underbrace{Re\,z\,Im\,w + Im\,z\,Re\,w}_{Im(zw)})$$

Notice $\bar{z} = Re(z) - i\,Im(z). \Rightarrow Im(\bar{z}) = -Im(z).$

$$\varphi(\bar{z}) = \begin{pmatrix} Re\,\bar{z} & -Im\,\bar{z} \\ Im\,\bar{z} & Re\,\bar{z} \end{pmatrix} = \begin{pmatrix} Re\,z & Im\,z \\ -Im\,z & Re\,z \end{pmatrix}$$

$$= \begin{pmatrix} Re\,z & -Im\,z \\ Im\,z & Re\,z \end{pmatrix}^T$$

$$= \varphi(z)^T$$

$\forall z, w \in \mathbb{C}.$

46) Let $\psi : gl(2,\mathbb{C}) \longrightarrow gl(4,\mathbb{R})$ be defined by

$A^i_j = A_{ij}$

$$\psi(A) = \begin{pmatrix} \phi(A^1_1) & \phi(A^1_2) \\ \phi(A^2_1) & \phi(A^2_2) \end{pmatrix}$$

Let $A^\dagger = (\bar{A})^T$ show $\psi(A^\dagger) = \psi(A)^T$

$$\psi(A^\dagger) = \left[\left(\phi\left[((\bar{A})^T)_{ij}\right]\right]\right.$$

$$= \left[\phi\left[(\bar{A})_{ji}\right]\right]$$

$$= \begin{bmatrix} \phi(\bar{A}_{11}) & \phi(\bar{A}_{21}) \\ \phi(\bar{A}_{12}) & \phi(\bar{A}_{22}) \end{bmatrix}$$

$$= \begin{bmatrix} (\phi(A_{11}))^T & (\phi(A_{21}))^T \\ (\phi(A_{12}))^T & (\phi(A_{22}))^T \end{bmatrix} \quad \text{by 4a.}$$

$$= \begin{bmatrix} \phi(A_{11}) & \phi(A_{12}) \\ \phi(A_{21}) & \phi(A_{22}) \end{bmatrix}^T$$

$$= \psi(A)^T$$

(4b) Continued $U(2) = \{ A \in gl(2, \mathbb{C}) \mid U^\dagger U = I \}$

$$\psi(U^\dagger) = \psi(U)^T$$

But $U^\dagger = U^{-1}$ since $U^\dagger U = I$.

Hence $\psi(U^{-1}) = \psi(U)^T$. It can be shown

1.) $\psi(AB) = \psi(A)\psi(B)$ (proof below.)

2.) $\psi(I_2) = I_4$. (clear)

$$I = \psi(I) = \psi(U^\dagger U) = \psi(U^\dagger)\psi(U)$$
$$= \psi(U)^T \psi(U)$$
$$\therefore \underline{\psi(U) \in O(4)}$$

$$\psi(A)\psi(B) = \begin{bmatrix} \phi(A_{11}) & \phi(A_{12}) \\ \phi(A_{21}) & \phi(A_{22}) \end{bmatrix} \begin{bmatrix} \phi(B_{11}) & \phi(B_{12}) \\ \phi(B_{21}) & \phi(B_{22}) \end{bmatrix}$$
$$= \begin{bmatrix} \phi(A_{11})\phi(B_{11}) + \phi(A_{12})\phi(B_{21}) & \text{---} \\ \text{---} & \text{---} \end{bmatrix}$$
$$= \begin{bmatrix} \phi(A_{11}B_{11} + A_{12}B_{21}) & \phi(A_{11}B_{12} + A_{12}B_{22}) \\ \phi(A_{21}B_{11} + A_{22}B_{21}) & \phi(A_{21}B_{12} + A_{22}B_{22}) \end{bmatrix}$$
$$= \psi(AB).$$

46.) Show $\psi(U) \in SO(4,\mathbb{R})$ we already showed $\psi(U) \in O(4,\mathbb{R})$ given $U^{\dagger}U = I$. Need to show

$$\det(\psi(U)) = 1.$$

Note that $\det(U^{\dagger}U) = \overline{\det(U)}\det(U)$

thus $|\det(U)|^2 = 1$ $\therefore$ $\det(U) \in S^1$

Where $S^1 \subset \mathbb{C}$ the complex unit circle.

We see $U(2)$ is connected, then as ψ is a continuous map it follows $\psi(U(2))$ is connected since $\psi(I) = I \Rightarrow \psi(U(2)) \subseteq \underbrace{SO(4,\mathbb{R})}_{\text{connected.}}$