Homework 1 on Linear Systems and Gaussian Elimination:

From Spence, Insel & Friedberg, Elementary Linear Algebra 2^{nd} Ed.

§1.3 # 4, 46 // §1.4 # 4, 8, 10, 48 // §1.5 # 28

§1.3#4 Write the coefficient matrix and augmented coeff matrix for system given below,

$$\begin{aligned} x_1 + 2x_3 - x_4 &= 3 \\ 2x_1 - 2x_2 + x_4 &= 0 \end{aligned} \iff Ax = b \quad \text{where}$$

$$\underbrace{A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & -2 & 0 & 1 \end{bmatrix}}_{\text{coeff. matrix}} \qquad [A|b] = \underbrace{\left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 3 \\ 2 & -2 & 0 & 1 & 0 \end{array}\right]}_{\text{aug. coeff. matrix.}}$$

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§1.3#46 Suppose $\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \text{rref}[A|b]$.

It follows that the system $Ax = b$ is inconsistent since if $x = [x_1, x_2, x_3]^T$ then the 3^{rd} row states $0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 1$ which is clearly false. $\therefore$ No sol^{n}s. for $Ax = b$.

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§1.4#4

$$\begin{aligned} x_1 - x_2 - 3x_3 &= 3 \\ 2x_1 + x_2 - 3x_3 &= 0 \end{aligned}$$

$$[A|b] = \left[\begin{array}{ccc|c} 1 & -1 & -3 & 3 \\ 2 & 1 & -3 & 0 \end{array}\right] \xrightarrow{r_2 - 2r_1 \to r_2} \left[\begin{array}{ccc|c} 1 & -1 & -3 & 3 \\ 0 & 3 & 3 & -6 \end{array}\right]$$

$$\xrightarrow{r_2/3 \to r_2} \left[\begin{array}{ccc|c} 1 & -1 & -3 & 3 \\ 0 & 1 & 1 & -2 \end{array}\right] \xrightarrow{r_1 + r_2 \to r_1} \underbrace{\left[\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -2 \end{array}\right]}_{\text{can read sol}^n\text{s from this easily.}} = \text{rref}[A|b]$$

Note x_3 is free, and

$$\boxed{\begin{aligned} x_1 &= 1 + 2x_3 \\ x_2 &= -2 - x_3 \\ x_3 &= x_3 \end{aligned}}$$

(it's consistent.)

§1.4#8

$$x_1 + x_2 - x_3 - x_4 = -2$$
$$2x_2 - 3x_3 - 12x_4 = -3$$
$$x_1 + x_3 + 6x_4 = 0$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -2 \\ 0 & 2 & -3 & -12 & -3 \\ 1 & 0 & 1 & 6 & 0 \end{array}\right] \xrightarrow{r_3 - r_1 \to r_3} \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -2 \\ 0 & 2 & -3 & -12 & -3 \\ 0 & -1 & 2 & 7 & 2 \end{array}\right]$$

$$\xrightarrow[r_2 + 2r_3 \to r_2]{r_1 + r_3 \to r_1} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 6 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & -1 & 2 & 7 & 2 \end{array}\right]$$

$$\xrightarrow[r_3 - 2r_2 \to r_3]{r_1 - r_2 \to r_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 4 & -1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & -1 & 0 & 3 & 0 \end{array}\right]$$

$$\xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 4 & -1 \\ 0 & -1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 2 & 1 \end{array}\right]$$

$$\xrightarrow{-r_2 \to r_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 4 & -1 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 2 & 1 \end{array}\right]$$

Thus the system is consistent, moreover x_4 is free and the general sol^n is $\boxed{\begin{array}{l} x_1 = -1 - 4x_4 \\ x_2 = 3x_4 \\ x_3 = 1 - 2x_4 \\ x_4 = x_4 \end{array}}$ for all $x_4 \in \mathbb{R}$.

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§1.4#10

$$\left.\begin{array}{r} x_1 - 3x_2 + x_3 + x_4 = 0 \\ -3x_1 + 9x_2 - 2x_3 - 5x_4 = 1 \\ 2x_1 - 6x_2 - x_3 + 8x_4 = -2 \end{array}\right\} \to \left[\begin{array}{cccc|c} 1 & -3 & 1 & 1 & 0 \\ -3 & 9 & -2 & -5 & 1 \\ 2 & -6 & -1 & 8 & -2 \end{array}\right] \xrightarrow[r_3 - 2r_1 \to r_3]{r_2 + 3r_1 \to r_2} \left[\begin{array}{cccc|c} 1 & -3 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & -3 & 6 & -2 \end{array}\right]$$

$$\xrightarrow[r_3 + 3r_2 \to r_3]{r_1 - r_2 \to r_1} \left[\begin{array}{cccc|c} 1 & -3 & 0 & 3 & -1 \\ 0 & 0 & 1 & -2 & 1 \\ \underbrace{0 \quad 0 \quad 0 \quad 0}_{} & & & & 1 \end{array}\right]$$

$\therefore$ this system is inconsistent, No sol^{n}s.

§1.4 #48 Find a polynomial function $f(x) = Ax^2 + Bx + C$ whose graph contains $(-2,-33)$, $(2,-1)$ and $(3,-8)$

Plug in the given data,

$$f(-2) = 4A - 2B + C = -33$$
$$f(2) = 4A + 2B + C = -1$$
$$f(3) = 9A + 3B + C = -8$$

$$\begin{array}{cccc} C & B & A & \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 4 & -33 \\ 1 & 2 & 4 & -1 \\ 1 & 3 & 9 & -8 \end{array}\right] \xrightarrow[r_3 - r_1 \to r_3]{r_2 - r_1 \to r_2} \left[\begin{array}{ccc|c} 1 & -2 & 4 & -33 \\ 0 & 4 & 0 & 32 \\ 0 & 5 & 5 & 25 \end{array}\right]$$

$$\xrightarrow[\frac{1}{5} r_3 \to r_3]{\frac{1}{4} r_2 \to r_2} \left[\begin{array}{ccc|c} 1 & -2 & 4 & -33 \\ 0 & 1 & 0 & 8 \\ 0 & 1 & 1 & 5 \end{array}\right] \xrightarrow[r_3 - r_2 \to r_3]{r_1 + 2r_2 \to r_1} \left[\begin{array}{ccc|c} 1 & 0 & 4 & -17 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

$$\xrightarrow{r_1 - 4r_3 \to r_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -3 \end{array}\right] \Rightarrow \boxed{\begin{array}{l} C = -5 \\ B = 8 \\ A = -3 \end{array}}$$

$$\therefore \boxed{f(x) = -3x^2 + 8x - 5}$$

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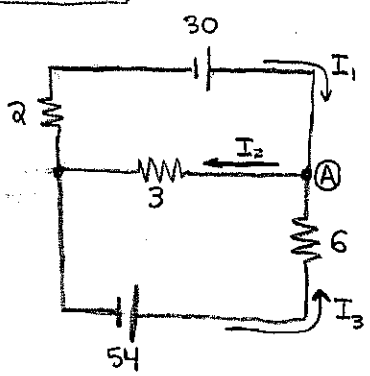
§1.5 #28 Determine I_1, I_2, I_3 in the circuit given below



top loop : $2I_1 - 30 + 3I_2 = 0$: Eq.ⁿ I

bottom loop : $54 - 3I_2 - 6I_3 = 0$: Eq.ⁿ II

current conservation at (A) : $I_1 - I_2 + I_3 = 0$: Eq.ⁿ III

I, II $\to$ Set $3I_2 = \underline{30 - 2I_1 = 54 - 6I_3}$ Eq.ⁿ IV

II, III $\to$ Set $I_2 = \underline{I_1 + I_3 = 18 - 2I_3}$ Eq.ⁿ V

Clean up IV & V

$$\begin{aligned} -24 &= 2I_1 - 6I_3 \\ 18 &= I_1 + 3I_3 \end{aligned} \Rightarrow \begin{aligned} -24 &= 2I_1 - 6I_3 \\ 36 &= 2I_1 + 6I_3 \end{aligned} \Rightarrow 4I_1 = 12 \Rightarrow \boxed{I_1 = 3}$$

Thus $I_3 = \frac{1}{3}(18 - I_1) = \frac{1}{3}(18-3) = \frac{1}{3}(15) = 5 \therefore \boxed{I_3 = 5}$

Return to Eq.ⁿ III, $I_2 = I_1 + I_3 = 3 + 5 = 8 \therefore \boxed{I_2 = 8}$

<u>Remark</u>: I omitted units, also this algebra is cluttered, sorry.