Homework 2 on Matrix Multiplication and Math.

From Spence, Insel & Friedberg, Elementary Linear Algebra 2nd Ed.
§1.2 # 6, 8, 12, 16 // §2.1 # 10, 12, 14, 16, 50, 71 // §2.3 # 20, 24, 59

§1.2 #6 Multiply, $(1\times 3)(3\times 1) \to (1\times 1)$ ← I don't require you write this. I write it to help me check my dimensions.

$$\begin{bmatrix} 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix} = -4+4+18 = \boxed{18.}$$

§1.2 #8 Multiply. Note $(3\times 3)(3\times 1) \to (3\times 1)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+0+0 \\ 0+b+0 \\ 0+0+c \end{bmatrix} = \boxed{\begin{bmatrix} a \\ b \\ c \end{bmatrix}.}$$

§1.2 #12 Multiply. $(3\times 2)(2\times 1) \to (3\times 1)$

$$\begin{bmatrix} 3 & -3 \\ -2 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0-3 \\ 0+4 \\ 0+2 \end{bmatrix} = \boxed{\begin{bmatrix} -3 \\ 4 \\ 2 \end{bmatrix}.}$$

§1.2 #16 Multiply. $(2\times 2)(2\times 1) \to (2\times 1)$

$$\left(\begin{bmatrix} 3 & 0 \\ -2 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & -3 \end{bmatrix}\right)\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 16+10 \\ 4+5 \end{bmatrix} = \boxed{\begin{bmatrix} 26 \\ 9 \end{bmatrix}.}$$

§2.1 #10 Let $A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$ and $z = [7, -1]$ we can calculate Az^T,

$$Az^T = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 7 \\ -1 \end{bmatrix} = \boxed{\begin{bmatrix} 9 \\ 17 \end{bmatrix}.}$$

§2.1 #12 Let $A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & 8 & 1 \\ 2 & 0 & 4 \end{bmatrix}$ we can calculate AC,

$$AC = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 & 8 & 1 \\ 2 & 0 & 4 \end{bmatrix} = \boxed{\begin{bmatrix} -1 & 8 & -7 \\ 21 & 24 & 19 \end{bmatrix}.}$$

§2.1 #14 yes BA defined below makes sense,

$$BA = \begin{bmatrix} 7 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} = \boxed{\begin{bmatrix} 19 & 2 \\ 7 & 6 \end{bmatrix}.}$$

§2.1#16 $C = \begin{bmatrix} 3 & 8 & 1 \\ 2 & 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 7 & 4 \\ 1 & 2 \end{bmatrix}$ the product CB is not defined since $(2\times 3)(2\times 2)$ does not match the definition for matrix multiplication. (yes you could insert extra zeros to make $\bar{C}\bar{B}$ well-defined, but that would not be the standard matrix multiplication)

§2.1#50 In a symmetric $n\times n$ matrix A we have $A^T = A$ thus $(A^T)_{ij} = A_{ij} \Rightarrow A_{ji} = A_{ij} \ \forall i,j$.

Remark: I meant to assign #51, oops! (See Homework 3 Sol$^{\underline{n}}$ for §2.1#51)

§2.1#71 Let $A = \left[\begin{array}{c|c} 0.85 & 0.03 \\ \hline 0.15 & 0.97 \end{array}\right]$ and suppose that $P = \begin{bmatrix} 500 \\ 700 \end{bmatrix}$ where $500 = \#$ of people in city, $700 = \#$ of people in suburbs. Generally $A^k P = \begin{bmatrix} \text{city after } k \text{ years} \\ \text{suburbs after } k \text{ years} \end{bmatrix}$

a.) $A^{10}P = \begin{bmatrix} 0.85 & 0.03 \\ 0.15 & 0.97 \end{bmatrix}^{10} \begin{bmatrix} 500 \\ 700 \end{bmatrix}$

$\cong \begin{bmatrix} 0.2812 & 0.1438 \\ 0.7188 & 0.8562 \end{bmatrix} \begin{bmatrix} 500 \\ 700 \end{bmatrix}$ ← TI-89.

$\cong \boxed{\begin{bmatrix} 241.23 \\ 958.77 \end{bmatrix}}$ ← city, ← suburbs.

b.) $A^{20}P = A^{10}A^{10}P$

$\cong \begin{bmatrix} 0.2812 & 0.1438 \\ 0.7188 & 0.8562 \end{bmatrix} \begin{bmatrix} 241.23 \\ 958.77 \end{bmatrix}$

$\cong \boxed{\begin{bmatrix} 205.67 \\ 994.33 \end{bmatrix}}$ ← city ← suburbs.

c.) $A^{50}P = \begin{bmatrix} 0.2812 & 0.1438 \\ 0.7188 & 0.8562 \end{bmatrix}^5 \begin{bmatrix} 500 \\ 700 \end{bmatrix} = \begin{bmatrix} 200.02 \\ 999.99 \end{bmatrix}$ ← city ← suburbs.

d.) I conjecture $A^k P \longrightarrow \begin{bmatrix} 200 \\ 1000 \end{bmatrix}$ as $k \longrightarrow \infty$.

$\underbrace{\text{(You can check that } Ax = x \text{ has sol}^{\underline{n}}}_{\text{not required, but see my notes.}} \begin{bmatrix} 200 \\ 1000 \end{bmatrix})$.

§2.3#20)

$$E_{4r_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} = [I : \underbrace{4r_3 \longrightarrow r_3}]$$

the inverse row operation is $r_3/4 \longrightarrow r_3$

$$\Rightarrow \boxed{E_{4r_3}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}.}$$

§2.3#24) Find elementary matrix E such that $EA = B$

$$A = \begin{bmatrix} -1 & 5 \\ 2 & 3 \end{bmatrix} \xrightarrow{r_2 \to 2r_1 + r_2} \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$$

Since $B = [A : r_2 \longrightarrow 2r_1 + r_2]$ we find

$$\underbrace{E_{r_2 \longrightarrow 2r_1 + r_2}}\ A = B$$

$$\boxed{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -1 & 5 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}.} \qquad \boxed{E = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.}$$

§2.3#59) Prove that if $A, B, C \in GL(n)$ then $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

By Prop. 2.5.5 it suffices to check

$$\begin{aligned}
(ABC)(C^{-1}B^{-1}A^{-1}) &= ABCC^{-1}B^{-1}A^{-1} && \text{: assoc. of mat. mult.} \\
&= ABIB^{-1}A^{-1} && \text{: def}^{\underline{n}} \text{ of } C^{-1} \\
&= ABB^{-1}A^{-1} && \text{: prop. of Identity matrix } I \\
&= AIA^{-1} && \text{: } B^{-1} \text{ def}^{\underline{n}}. \\
&= AA^{-1} && \text{: prop. of } I. \\
&= I. && \text{: def}^{\underline{n}} \text{ of } A^{-1}
\end{aligned}$$

$$\therefore \boxed{(ABC)^{-1} = C^{-1}B^{-1}A^{-1}}$$

(we gave full credit for some shorter answers, but this answer is tediously complete)