HOMEWORK 4 : SPANNING, LINEAR INDEPENDENCE, BASES, CCP etc...

FROM SPENCE, INSEL & FRIEDBERG // Elementary Linear Algebra 2nd Ed.
§1.6 # 4, 26, 30, 44, 69 // §1.7 # 24, 36 // §2.3 # 68, 82 // §4.4 # 14, 28 // §7.1 # 31 // §7.3 # 4

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§1.6 #4 Is $[2,-1,3]^T \in \text{span}\left\{ \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\3\end{bmatrix} \right\}$?

In other words, do $\exists\, x, y, z \in \mathbb{R}$ such that

$$x\begin{bmatrix}1\\0\\1\end{bmatrix} + y\begin{bmatrix}-1\\1\\1\end{bmatrix} + z\begin{bmatrix}1\\1\\3\end{bmatrix} = \begin{bmatrix}2\\-1\\3\end{bmatrix} \Rightarrow \begin{bmatrix}1 & -1 & 1\\0 & 1 & 1\\1 & 1 & 3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}2\\-1\\3\end{bmatrix}$$

We know how to solve such problems (if it's consistent !)

$$\text{rref}\left[\begin{array}{ccc|c}1 & -1 & 1 & 2\\0 & 1 & 1 & -1\\1 & 1 & 3 & 3\end{array}\right] = \left[\begin{array}{ccc|c}1 & 0 & 2 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 1\end{array}\right]$$ ← inconsistent system notice the 3rd row.

$\therefore$ $\underline{[2,-1,3]^T \text{ is not in the span.}}$

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§1.6 #26 Is $\left\{ \begin{bmatrix}-1\\2\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\3\end{bmatrix}, \begin{bmatrix}1\\-3\\1\end{bmatrix} \right\}$ a generating set for $\mathbb{R}^{3\times 1}$?

Observe $\text{rref}\begin{bmatrix}-1 & -1 & 1\\2 & 1 & -3\\1 & 3 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & -2\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix}$, thus $A = \begin{bmatrix}-1 & -1 & 1\\2 & 1 & -3\\1 & 3 & 1\end{bmatrix}$ is $\underline{\text{not}}$ invertible $\Rightarrow \exists\, \vec{b} \in \mathbb{R}^{3\times 1}$ such that $A\vec{x} = \vec{b}$ has no sol$^{\underline{n}}$.

Let me add a bit more to this sol$^{\underline{n}}$, Let's find out which vectors in $\mathbb{R}^{3\times 1}$ are generally not in the span. In other words find $\vec{B}$ such that $A\vec{x} = \vec{B}$ is inconsistent, $\vec{B} = [a,b,c]^T$

$$A\vec{x} = \vec{B} \Leftrightarrow \left[\begin{array}{ccc|c}-1 & -1 & 1 & a\\2 & 1 & -3 & b\\1 & 3 & 1 & c\end{array}\right] \xrightarrow[r_3+r_1]{r_2+2r_1} \left[\begin{array}{ccc|c}-1 & -1 & 1 & a\\0 & -1 & -1 & b+2a\\0 & 2 & 2 & c+a\end{array}\right] \xrightarrow{r_3+2r_2} \left[\begin{array}{ccc|c}-1 & -1 & 1 & a\\0 & -1 & -1 & b+2a\\0 & 0 & 0 & c+a+2(b+2a)\end{array}\right]$$

Now we could continue row-reductions until we found rref(A) but there's no need. We can see that if $c+a+2(b+2a) \neq 0$ then the system $A\vec{x} = \vec{B}$ will be inconsistent. Consider,

$$c + a + 2(b+2a) = 5a + 2b + c = 1$$ ← a choice, just for discussion's sake.

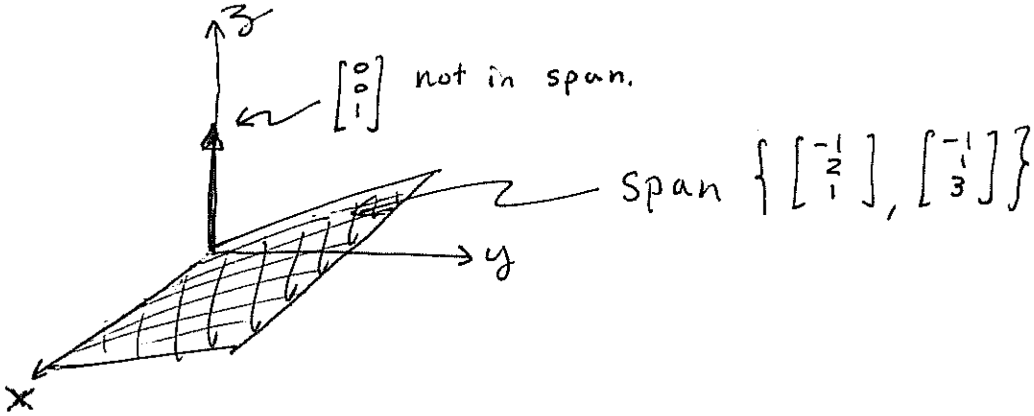
choose $a = b = 0$.

$$\Rightarrow \underline{\begin{bmatrix}0\\0\\1\end{bmatrix} \notin \text{span}\left\{ \begin{bmatrix}-1\\2\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\3\end{bmatrix}, \begin{bmatrix}1\\-3\\1\end{bmatrix} \right\}}$$

continued ↷

Extra comment continued

I just gave evidence that $\begin{bmatrix}0\\0\\1\end{bmatrix} \notin \text{span}\left\{\begin{bmatrix}-1\\2\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\3\end{bmatrix}, \begin{bmatrix}1\\-3\\1\end{bmatrix}\right\}$.



As you can see from this heuristic picture there are many vectors which are not in the plane spanned by the given generating set.

§1.6 #30 Is $A\vec{x} = \vec{B}$ consistent for all $b \in \mathbb{R}^{2\times 1}$ given $A = \begin{bmatrix}1 & -2\\2 & -4\end{bmatrix}$?

Many sol$^{\underline{n}}$'s are possible. Here's one, note

$$\det(A) = \det\begin{bmatrix}1 & -2\\2 & -4\end{bmatrix} = -4+4 = 0$$

$\therefore A^{-1}$ does not exist
$\Rightarrow \exists \vec{B}$ such that $A\vec{x} = \vec{B}$ is inconsistent.

Therefore, to answer the given question, NO.

(I again illustrate how to find vectors not in the span)

This was not asked for, but I'll again show how to find the b which is not mapped to by Ax,

$$[A|\vec{B}] = \left[\begin{array}{cc|c}1 & -2 & a\\2 & -4 & b\end{array}\right] \xrightarrow{r_2 - 2r_1} \left[\begin{array}{cc|c}1 & -2 & a\\0 & 0 & b-2a\end{array}\right]$$

any choice of a, b for which $b - 2a \neq 0$ gives inconsistent system.

For example, $\vec{B} = \begin{bmatrix}0\\1\end{bmatrix}$ makes $A\vec{x} = \vec{B}$ inconsistent.

§1.6 #44 Let $S = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$ find a subset $\tilde{S}$ of S for which $\text{span}(S) = \text{span}(\tilde{S})$

I'll use the CCP and associated wisdom, the principle observation is that the pivot columns are LI, calculate then

$$\text{rref}\underbrace{\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}}_{\substack{\text{pivot}\\\text{columns}\\\text{of } A}}\!\!\begin{bmatrix}0\\0\\1\end{bmatrix} = \underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}}_{\text{pivot cols.}}\!\!\begin{bmatrix}0\\-1\\1\end{bmatrix} = R$$

note then, by CCP,

$$\underbrace{\begin{bmatrix}0\\-1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix} - \begin{bmatrix}0\\1\\0\end{bmatrix}}_{\substack{\text{from } R\\ col_4(R) = col_3(R) - col_2(R)}} \Rightarrow \underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}1\\1\\0\end{bmatrix}}_{\substack{\text{same linear}\\\text{combo. works}\\\text{for } A.}}$$

Clearly $col_4(A) = \begin{bmatrix}0\\0\\1\end{bmatrix}$ is redundant and we can use $\tilde{S} = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}$.

§1.6 #69 Let S_1, S_2 be finite subsets of $\mathbb{R}^{n\times 1}$ such that $S_1 \subset S_2$. Prove that if S_1 is a generating set of $\mathbb{R}^{n\times 1}$ then so is S_2

We are given that each $V \in \mathbb{R}^{n\times 1}$ is a linear combination of the vectors in S_1 and also $S_1 \subset S_2$. Let us denote $S_1 = \{V_1, V_2, \dots, V_k\}$. If $W \in \mathbb{R}^{n\times 1}$ then $\exists\, W_1, W_2, \dots, W_k \in \mathbb{R}$ such that $W = W_1V_1 + W_2V_2 + \cdots + W_kV_k$. Notice that $V_i \in S_1 \Rightarrow V_i \in S_2$ for each $i = 1, 2, \dots, k$ since $S_1 \subset S_2$. Hence $W \in \text{span}(S_2)$ since $W = W_1V_1 + \cdots + W_kV_k$ is a linear combination of vectors in S_2. Thus every vector in $\mathbb{R}^{n\times 1}$ is a linear combination of vectors in S_2. It follows that S_2 is a generating set for $\mathbb{R}^{n\times 1}$!

§1.7#24 Is $S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$ a LI set?

Note that $\text{rref}\begin{bmatrix} 1 & -1 & 2 \\ -1 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ thus S is LI since we know the pivot columns of a matrix are LI.

§1.7#36 Write a vector in S as a linear combination of other vectors in S given that $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \\ -5 \end{bmatrix} \right\}$

We'll use the CCP to do this efficiently. Note

$$\text{rref}(A) = \text{rref}\begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & -4 \\ 3 & 1 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} = R$$

Notice $col_3(R) = -3\,col_1(R) + 4\,col_2(R)$ (you can just see this by inspection!) then by CCP we know

$$col_3(A) = -3\,col_1(A) + 4\,col_2(A) \quad \text{or} \quad \underline{\begin{bmatrix} 5 \\ -4 \\ -5 \end{bmatrix} = -3\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + 4\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}}.$$

Remark: other answers are possible, however they just amount to rewriting the underlined eqn above.

§2.3#68 Find A given that $\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$ and $col_1(A) = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$, $col_3(A) = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$, $col_5(A) = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$

Again the CCP is key. Observe that:

$col_2(R) = 2\,col_1(R)$, $col_4(R) = -3\,col_1(R) + 2\,col_3(R)$, $col_6(R) = col_1(R) + 2\,col_3(R) + 3\,col_5(R)$

Then the same correspondences hold for columns of A so we find,

$$A = \left[\begin{array}{c|c|c|c|c|c} 2 & 4 & 1 & -6+2 & 2 & 2+2+6 \\ 0 & 0 & -1 & -2 & 3 & 0-2 \\ -1 & -2 & 2 & 3+4 & 0 & -1+4+0 \\ 1 & 2 & 0 & -3 & 1 & 1+0+3 \end{array}\right] = \underline{\left[\begin{array}{c|c|c|c|c|c} 2 & 4 & 1 & -4 & 2 & 10 \\ 0 & 0 & -1 & -2 & 3 & 7 \\ -1 & -2 & 2 & 7 & 0 & 3 \\ 1 & 2 & 0 & -3 & 1 & 4 \end{array}\right]}$$

given

generated from CCP.

⑤

§2.3#82 Given $B = \begin{bmatrix} 1 & 0 & 1 & -3 & -1 & 4 \\ 2 & -1 & 3 & -8 & -1 & 9 \\ -1 & 1 & -2 & 5 & 1 & -6 \\ 0 & 1 & -1 & 2 & 1 & -3 \end{bmatrix}$ write $col_6(B)$ as a linear combination of the pivot columns of B.

We can calculate,

$$\text{rref}(B) = \begin{bmatrix} 1 & 0 & 1 & -3 & 0 & 3 \\ 0 & 1 & -1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \underline{col_6(B) = 3col_1(B) - 2col_2(B) - col_5(B}$$

$$\therefore \begin{bmatrix} 4 \\ 9 \\ -6 \\ -3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}.$$

(oops, §4.4#14 is on ⑥)

§4.4# Find unique representation of $\vec{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ as linear combination of $b_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $b_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $b_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

We wish to solve $\vec{u} = xb_1 + yb_2 + zb_3$, use our usual notation,

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ -1 & 0 & 1 & b \\ 2 & 2 & 1 & c \end{array}\right]$$

$$\xrightarrow[r_3 - 2r_1]{r_2 + r_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & 1 & a+b \\ 0 & 0 & 1 & c-2a \end{array}\right] \xrightarrow{r_1 - r_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & a-(a+b) \\ 0 & 1 & 1 & a+b \\ 0 & 0 & 1 & c-2a \end{array}\right]$$

$$\xrightarrow[r_2 - r_3]{r_1 + r_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -b+c-2a \\ 0 & 1 & 0 & a+b-(c-2a) \\ 0 & 0 & 1 & c-2a \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2a-b+c \\ 0 & 1 & 0 & 3a+b-c \\ 0 & 0 & 1 & -2a+c \end{array}\right]$$

Thus we read $x = -2a-b-c$, $y = 3a+b-c$, $z = -2a+c$ and

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = (-2a-b+c)\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + (3a+b-c)\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + (-2a+c)\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Remark: we just found $[v]_\beta = [-2a-b+c,\ 3a+b-c,\ -2a+c]^T$ given $v = [a,b,c]^T$ and $\beta = \{b_1, b_2, b_3\}$. Also I believe you can check $[b_1|b_2|b_3]^{-1} = \begin{bmatrix} -2 & -1 & 1 \\ 3 & 1 & -1 \\ -2 & 0 & 1 \end{bmatrix}$.

§4.4 #14

(a.) Prove $\beta = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^{3\times 1}$

(b.) Find $[V]_\beta$ given $V = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} - 4\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(a.) I'll be lazy, $\det \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = 1(0) + 1(-2) + 1(-2) = -4 \neq 0$

thus $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ is invertible and $Ax = b$ is consistent for every $b \in \mathbb{R}^{3\times 1}$ so $\text{span}\,\beta = \mathbb{R}^{3\times 1}$. Linear independence follows for a variety of reasons. I'll name two

(i) – if β had a linear dependence then A would have linear dep. columns $\Rightarrow \det(A) = 0$ BUT, we just calculated $\det(A) \neq 0$ so we find a contradiction and can conclude β is LI.

(ii.) – $A\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0 \Rightarrow A^{-1}A\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = A^{-1}0 \Rightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0$

Hence, by prop. which gave equivalent def^{n} of LI, we find β is LI since

$A\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0 \Rightarrow c_1 = c_2 = c_3 = 0.$

(b.) $V = 1\cdot\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + 0\cdot\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} - 4\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$= [\beta][V]_\beta$ if $\underline{[V]_\beta = [1, 0, -4]^T}$.

$\boxed{\S 7.1 \# 31}$ Prove $\text{span}\{1+x, 1-x, 1+x^2, 1-x^2\} = P_2$

It is immediately obvious $\text{span}\{1+x, 1-x, 1+x^2, 1-x^2\} \subseteq P_2$. However, the reverse inclusion requires some calculation. Let $ax^2+bx+c \in P_2$ where $a, b, c \in \mathbb{R}$, note

$$
\begin{aligned}
ax^2+bx+c &= a\left[\tfrac{1}{2}(1+x^2) - \tfrac{1}{2}(1-x^2)\right] + \\
&\quad + b\left[\tfrac{1}{2}(1+x) - \tfrac{1}{2}(1-x)\right] + \\
&\quad + c\left[\tfrac{1}{2}\left[(1+x) + (1-x)\right]\right] \in \text{span}\{1+x, 1-x, 1+x^2, 1-x^2\}
\end{aligned}
$$

$$\therefore \; P_2 \subseteq \text{span}\{1+x, 1-x, 1+x^2, 1-x^2\}$$

We conclude $\underline{P_2 = \text{span}\{1+x, 1-x, 1+x^2, 1-x^2\}}$.

<u>Remark:</u> You might wonder how did I see that $x^2 = \frac{1}{2}(1+x^2) - \frac{1}{2}(1-x^2)$ and $x = \frac{1}{2}(1+x) - \frac{1}{2}(1-x)$ and $1 = \frac{1}{2}(1+x) + \frac{1}{2}(1-x)$? Personally, I can just see it by inspection in this case. So what if you don't just "see" it, how would you go about solving this? You try to solve

$$ax^2+bx+c = c_1(1+x) + c_2(1-x) + c_3(1+x^2) + c_4(1-x^2)$$

for arbitrary a, b, c. This gives

$$\left.\begin{aligned}
a &= c_3 + c_4 \\
b &= c_1 - c_2 \\
c &= c_1 + c_2 + c_3 + c_4
\end{aligned}\right\} \text{ these can be solved in many ways.}$$

$\boxed{\S 7.3\#4}$ Is $S = \left\{ \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \right\}$ LI ?

Assume that:

$$c_1 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad (*)$$

$$\Rightarrow \left[\begin{array}{c|c} c_1 + c_2 + c_3 & 2c_1 + 3c_2 + 2c_3 \\ \hline 2c_1 + 3c_2 + 3c_3 & c_1 + c_2 + c_3 \end{array} \right] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} c_1 + c_2 + c_3 &= 0 \\ 2c_1 + 3c_2 + 2c_3 &= 0 \\ 2c_1 + 3c_2 + 3c_3 &= 0 \\ c_1 + c_2 + c_3 &= 0 \end{aligned}$$

redundant, just repeats of the eq^{n}'s from the (1,1) or

$$\begin{array}{ccc} c_1 & c_2 & c_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 3 & 2 & 0 \\ 2 & 3 & 3 & 0 \end{array} \right] \xrightarrow[r_3 - 2r_1]{r_2 - 2r_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right]$$

$$\xrightarrow[r_3 - r_2]{r_1 - r_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

$$\Rightarrow \text{rref} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 3 & 2 & 0 \\ 2 & 3 & 3 & 0 \end{array} \right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \Rightarrow \underline{\begin{aligned} c_1 &= 0 \\ c_2 &= 0 \\ c_3 &= 0 \end{aligned}}.$$

Thus S is a linearly independent set.

(I'm using Proposition 4.4.3 in my notes which is often a convenient criteria to test for LI)