Homework 7: Dot-Products & Orthogonality

§ 6.1 # 4, 14, 38, 52, 87 // §6.2 # 14, 22 // §6.3 # 2, 6 // §6.2 #63

§6.1#4 Let $u = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$ compute the norms of u & v and find the distance between u & v.

$$\|u\| = \sqrt{u \cdot u} = \sqrt{1^2+3^2+1^2} = \boxed{\sqrt{11} = \|u\|}.$$

$$\|v\| = \sqrt{v \cdot v} = \sqrt{1^2+4^2+2^2} = \boxed{\sqrt{21} = \|v\|}.$$

$$d(u,v) = \|u-v\| = \|[2,-1,-1]^T\| = \sqrt{2^2+1^2+1^2} = \boxed{\sqrt{6} = d(u,v)}.$$

§6.1#14 Let $u = \begin{bmatrix} 1 \\ 2 \\ -3 \\ -1 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 0 \end{bmatrix}$. Are u & v orthogonal?

If $u \cdot v = 0$ then we would say u & v are orthogonal, consider

$$u \cdot v = [1 \ 2 \ -3 \ -1]\begin{bmatrix} 2 \\ 3 \\ 2 \\ 0 \end{bmatrix} = 2+6-6+0 = 2 \neq 0$$

Thus u, v are not orthogonal since $\boxed{u \cdot v = 2}$.

§6.1#38 Let $u = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$. Find $\|u\|, \|v\|$ and $u \cdot v$. Illustrate the Cauchy Schwarz inequality with these vectors.

$$\|u\| = \sqrt{u \cdot u} = \sqrt{2}$$

$$\|v\| = \sqrt{v \cdot v} = \sqrt{4+1+9} = \sqrt{14}$$

$$u \cdot v = [0,1,1]\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} = 1+3 = 4$$

Note $\|u\|\|v\| = \sqrt{2}\sqrt{14} = \sqrt{28} > 4 = |u \cdot v| \ \therefore \ \underline{|u \cdot v| < \|u\|\|v\|}$

§6.1#52 We're given $u, v, w \in \mathbb{R}^{n \times 1}$ such that $\|u\| = 2$, $\|v\| = 3$, $\|w\| = 5$, $u \cdot v = -1$, $u \cdot w = 1$ and $v \cdot w = -4$. Calculate $(u+w) \cdot v$

$$(u+w) \cdot v = u \cdot v + w \cdot v = 1 - 4 = \boxed{-3.}$$

§6.1 #87 Let $\{v, w\}$ be a basis for a subspace $W \leq \mathbb{R}^{n\times 1}$ and define $z = \text{Proj}_v(w) = w - \left(\frac{v\cdot w}{v\cdot v}\right)v$. Prove that $\{v, z\}$ is an orthogonal basis for W

Observe that $\dim(W) = 2$ since $\{v, w\}$ is a basis for W. Thus if we can show $\{v, z\}$ is LI it follows that $\{v, z\}$ is a basis. But, we could also attack LI by an indirect argument. If we can show $\{v, z\}$ is a set of nonzero orthogonal vectors then we proved that LI follows. Consider then that

1.) $v \neq 0$ since otherwise $\{v, w\}$ would not be LI which contradicts the given fact that $\{v, w\}$ is a basis for W.

2.) $z \neq 0$ since otherwise $z = 0 \Rightarrow w - \left(\frac{v\cdot w}{v\cdot v}\right)v = 0$ which again implies linear dependence of $\{v, w\}$ which contradicts the given fact $\{v, w\}$ is a basis.

Therefore, $\{v, z\}$ is a set of nonzero vectors,

$$
\begin{aligned}
v \cdot z &= v \cdot \left[w - \left(\frac{v\cdot w}{v\cdot v}\right)v\right] \\
&= v\cdot w - \left(\frac{v\cdot w}{v\cdot v}\right) v\cdot v \\
&= v\cdot w - v\cdot w \\
&= 0
\end{aligned}
$$

$\therefore$ $\{v, z\}$ is an orthogonal set

$\Rightarrow$ $\{v, z\}$ is basis for W.

§6.2 #14 Let $S' = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ 5 \end{bmatrix} \right\}$.

$\qquad\qquad u_1 \qquad u_2 \qquad u_3$

a.) use Gram-Schmidt algorithm to find orthogonal set of vectors $\{V_1, V_2, V_3\}$ which spans span(S').

b.) normalize $\{V_1, V_2, V_3\}$ to get <u>orthonormal</u> basis for span(S')

a.) Apply Gram-Schmidt:

$$V_1 = u_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$

$$V_2 = u_2 - \left(\frac{u_2 \cdot V_1}{V_1 \cdot V_1}\right) V_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix} - \frac{6}{6}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

$$V_3 = u_3 - \left(\frac{u_3 \cdot V_2}{V_2 \cdot V_2}\right) V_2 - \left(\frac{u_3 \cdot V_1}{V_1 \cdot V_1}\right) V_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 5 \end{bmatrix} - \frac{8}{6}\begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix} - \frac{12}{6}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$

$$\Rightarrow V_3 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 5 \end{bmatrix} - \frac{4}{3}\begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ -2 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{4}{3}\begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/3 \\ -1/3 \\ -1/3 \end{bmatrix}$$

$$\boxed{\therefore \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1/3 \\ -1/3 \\ -1/3 \end{bmatrix} \right\} \leftarrow \text{orthogonal set}}$$

b.) normalize the vectors by dividing by their lengths

$$\boxed{\left\{ \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{12}}\begin{bmatrix} 3 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\}}$$

§6.2 #22 Find the linear combination of

$$V_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ -1 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad V_3 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \text{ to give } V = \begin{bmatrix} 6 \\ 9 \\ 9 \\ 0 \end{bmatrix}$$

Notice that $V_1 \cdot V_2 = 0$, $V_1 \cdot V_3 = 0$, $V_2 \cdot V_3 = 0$.
Let $V = aV_1 + bV_2 + cV_3$, we wish to calculate a, b, c.

$$V \cdot V_1 = aV_1 \cdot V_1 + bV_2 \cdot V_1 + cV_3 \cdot V_1 = aV_1 \cdot V_1 = 6a$$

$$\Rightarrow 6 - 18 = 6a \Rightarrow a = \frac{-12}{6} = -2 \quad \therefore \underline{a = -2}.$$

Likewise,

$$V \cdot V_2 = bV_2 \cdot V_2 \Rightarrow 30 = b(6)$$
$$\Rightarrow \underline{b = 5}.$$

$$V \cdot V_3 = cV_3 \cdot V_3 \Rightarrow 6 - 18 = c(6)$$
$$\Rightarrow -12 = 6c \quad \therefore \underline{c = -2}.$$

Thus, $\boxed{V = -2V_1 + 5V_2 - 2V_3}$

$$\begin{bmatrix} 6 \\ 9 \\ 9 \\ 0 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -2 \\ 0 \\ -1 \end{bmatrix} + 5\begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Remark: with orthogonal generating sets we can avoid row-reduction and instead take dot-products.

§6.3 #2 Let $S = \{[1, 0, 2]^T\}$ find basis for $S^\perp$

Let $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in S^\perp$ then $[1, 0, 2]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$, thus

$$x + 2z = 0 \Rightarrow \underline{x = -2z}$$

Use y & z as free variables,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2z \\ y \\ z \end{bmatrix} = y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \quad \therefore \boxed{S^\perp = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right\}}$$

Remark: this is same as calculating basis for $\text{Null}\left(\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}^T\right)$.

$\boxed{\S 6.3 \# 6}$ Given $S' = \{[1,-1,-5,-1]^T, [2,-1,-7,0]^T\}$ find basis for $S^\perp$

If $[x,y,z,w] \in S^\perp$ then $[x,y,z,w] \cdot [1,-1,-5,-1] = 0$ and $[x,y,z,w] \cdot [2,-1,-7,0] = 0$. This amounts to finding $[x,y,z,w]^T \in \text{Null}\left(\begin{bmatrix} 1 & 2 \\ -1 & -1 \\ -5 & -7 \\ -1 & 0 \end{bmatrix}^T\right)$. We could row reduce etc...
but these eq^{n}s are simple enough that there's no need for such sophistication,

$$\begin{aligned} x - y - 5z - w &= 0 \\ 2x - y - 7z &= 0 \end{aligned} \qquad \begin{aligned} x - 2z + w &= 0 \\ y &= x - 5z - w \end{aligned}$$

Use z & w as free variables,

$$\begin{aligned} x &= 2z - w \\ y &= x - 5z - w = 2z - w - 5z - w = -3z - 2w \\ z &= z \\ w &= w \end{aligned}$$

It was clear there should be <u>two</u> free varibles once it was clear that the vectors in S' were not linearly dependent, my intuition is guided by the Th$^{\underline{m}}$ $\text{Span}(S) \oplus S^\perp = \mathbb{R}^{4\times 1}$ if $\dim(\text{Span}(S')) = 2$ then $\dim(S^\perp) = 4 - 2 = 2 \Rightarrow S^\perp$ has basis with 2-vectors,

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2z - w \\ -3z - 2w \\ z \\ w \end{bmatrix} = z\begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} + w\begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \boxed{S^\perp = \text{span}\left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

<u>Remark</u>: Again, this is the same calculation as finding a basis for $\text{Null}([S]^T)$. Furthermore, you can check the answer by taking dot-products of vectors in $S^\perp$ with those in S'.

§6.2 # 63 Let $Q \in \mathbb{R}^{n \times n}$.
Prove $\{\mathrm{col}_i(Q)\}_{i=1}^{n}$ is an orthonormal basis for $\mathbb{R}^{n \times 1}$ iff $Q^T Q = I_n$

Proof: Let $Q \in \mathbb{R}^{n \times n}$.

$\{\mathrm{col}_i(Q)\}_{i=1}^{n}$ is an orthonormal basis for $\mathbb{R}^{n \times 1}$ $\Longleftrightarrow$

$$\Longleftrightarrow \quad \mathrm{col}_i(Q) \cdot \mathrm{col}_j(Q) = \delta_{ij}, \quad \forall i,j \in \mathbb{N}_n.$$

$$\Longleftrightarrow \quad (\mathrm{col}_i(Q))^T \mathrm{col}_j(Q) = \delta_{ij}, \quad \forall i,j \in \mathbb{N}_n.$$

$$\Longleftrightarrow \quad \mathrm{row}_i(Q^T)\, \mathrm{col}_j(Q) = \delta_{ij}, \quad \forall i,j \in \mathbb{N}_n.$$

$$\Longleftrightarrow \quad (Q^T Q)_{ij} = \delta_{ij}, \quad \forall i,j \in \mathbb{N}_n.$$

$$\Longleftrightarrow \quad Q^T Q = I_n.$$

(The problem is primarily a challenge of notation.)

Remark: If $Q^T Q = I_n$ then we say Q is an orthogonal matrix. ~~then~~
These are interesting because if we start with an orthonormal basis $\beta = \{v_1, v_2, \dots, v_n\}$ then $\beta' = \{Qv_1, Qv_2, \dots, Qv_n\}$ will also be an orthonormal basis. There will be a problem or two in the Problem Set to explore this remark further.