Homework 8: Least Squares & Orthogonality

§6.5 # 39, 42 // §6.4 # 2, 16, 38

§6.5#39 Let $0 < \theta < \pi$ and suppose $T: \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$T(e_1) = \cos\theta\, e_1 + \sin\theta\, e_2$$
$$T(e_2) = -\sin\theta\, e_1 + \cos\theta\, e_2$$
$$T(e_3) = e_3$$

(a.) prove T orthogonal, (b.) find e-values/vectors for T (c) describe T geometrically

(a.) $$\underbrace{T(x)\cdot T(y) = x\cdot y \quad \forall x,y \in \mathbb{R}^3}_{\text{orthogonal transformation}} \iff \underbrace{[T]^T[T] = I}_{\text{orthogonal matrix}}$$

We can check $[T]$ (the standard matrix of T)

$$[T]^T[T] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 & 0 \\ 0 & \cos^2\theta + \sin^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \therefore \ [T] \text{ is orthogonal matrix hence } T \text{ is orthogonal.}$$

(b.) Find e-values from characteristic eq$^{\underline{n}}$: note $T(v) = \lambda v$ iff $[T]v = \lambda v$ iff $([T] - \lambda I)v = 0$ (can use [T] to analyze T)

$$\det\left([T] - \lambda I\right) = \det\begin{bmatrix} \cos\theta - \lambda & +\sin\theta & 0 \\ -\sin\theta & \cos\theta - \lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda)\left[(\cos\theta - \lambda)^2 + \sin^2\theta\right]$$

$$\Rightarrow \underbrace{\lambda_1 = 1}_{\text{real e-value.}} \text{ or } \underbrace{\lambda_2 = \cos\theta \pm i\sin\theta = e^{\pm i\theta}}_{\text{complex e-value.}}$$

$\lambda_1 = 1$ Find $u_1 = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$ such that $([T]-I)u_1 = 0$

$$\begin{bmatrix} \cos\theta - 1 & +\sin\theta & 0 \\ -\sin\theta & \cos\theta - 1 & 0 \\ 0 & 0 & 1-1 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(\cos\theta - 1)u + \sin\theta\, v = 0 \longrightarrow \sin\theta(\cos\theta-1)u + \sin^2\theta\, v = 0$$
$$-\sin\theta\, u + (\cos\theta - 1)v = 0 \longrightarrow \underbrace{-\sin\theta(\cos\theta-1)u + (\cos\theta-1)^2 v = 0}_{\text{add these equations to find}}$$

$$(\sin^2\theta)v + (\cos\theta-1)^2 v = 0$$

$$v(\sin^2\theta + (\cos\theta-1)^2) = 0$$

$$\Rightarrow v = 0 \quad (\text{since } \theta \neq 0, \pi)$$

$$\Rightarrow u = 0$$

Hence, $u_1 = \begin{bmatrix} 0 \\ 0 \\ w \end{bmatrix}$ for $w \in \mathbb{R}$. Can choose $u_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ for applications.

$\lambda_2 = e^{i\theta}$ Find $u_2 = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \in \mathbb{C}^{3\times 1}$ such that $([T] - e^{i\theta}I)u_2 = 0$,

$$\begin{bmatrix} \cos\theta - e^{i\theta} & \sin\theta & 0 \\ -\sin\theta & \cos\theta - e^{i\theta} & 0 \\ 0 & 0 & 1-e^{i\theta} \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Notice $e^{i\theta} = \cos\theta + i\sin\theta$ and since $0 < \theta < \pi$ it follows $e^{i\theta} \neq 1$ hence $1 - e^{i\theta} \neq 0$ $\therefore$ $(1-e^{i\theta})w = 0$ yields $\underline{w = 0}$. In contrast, u & v will be nonzero since

$$\cos\theta - e^{i\theta} = \cos\theta - (\cos\theta - i\sin\theta) = i\sin\theta. \text{ Thus}$$

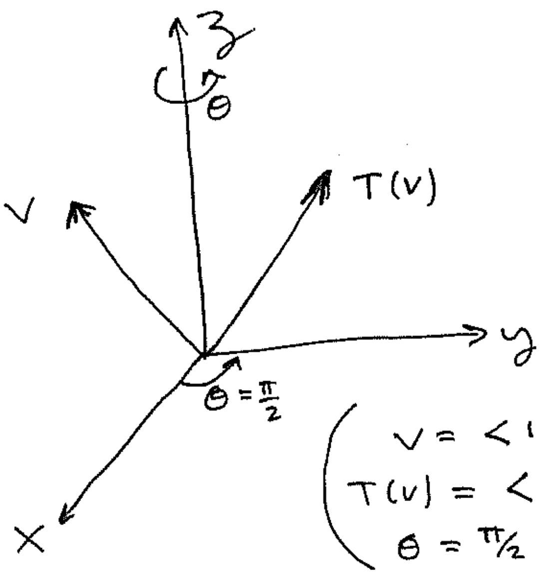
$$\left.\begin{aligned} i\sin\theta\, u + \sin\theta\, v &= 0 \\ -\sin\theta\, u + i\sin\theta\, v &= 0 \end{aligned}\right\} \longrightarrow u = iv \quad (\text{since } \sin\theta \neq 0)$$

$$\therefore \quad u_2 = \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(I chose $v = 1$ just to show you an example)

§ 6.5 #39 Continued

(c.) geometrically T is a rotation about the z-axis. Notice vectors along the z-axis are invariant however vectors off the z-axis are rotated by an angle θ.



$$\left(\begin{array}{l} v = \langle 1, 0, 2 \rangle^T \\ T(v) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \\ \theta = \pi/2 \text{ for example picture} \end{array} \right)$$

§ 6.5 #42 Show that $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined below is orthogonal

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Notice that $[T] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ thus we can deduce that T is orthogonal from the following calculation,

$$[T]^T[T] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Thus $[T]$ is an orthogonal matrix which implies T is an orthogonal operator. <u>In fact T is a reflection</u>.

§ 6.4 #2 Find the eqⁿ of the least squares line for the data: (1,30), (2,27), (4,21), (7,14)

We seek to find c_1, c_2 such that $y = c_1 x + c_2$ has a graph which is closest to the given set of points. We find that plugging in data yields:

$$\begin{aligned} 30 &= c_1 + c_2 \\ 27 &= 2c_1 + c_2 \\ 21 &= 4c_1 + c_2 \\ 14 &= 7c_1 + c_2 \end{aligned}$$

Which gives us the following matrix eq^{n}:

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 7 & 1 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{c} = \underbrace{\begin{bmatrix} 30 \\ 27 \\ 21 \\ 14 \end{bmatrix}}_{b}$$

We proved in lecture that the sol^{n} z to $M^T M z = M^T b$ gives best approx. to c in eq^{n} above.

$$M^T M = \begin{bmatrix} 1 & 2 & 4 & 7 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 7 & 1 \end{bmatrix} = \begin{bmatrix} 70 & 14 \\ 14 & 4 \end{bmatrix} \Rightarrow (M^T M)^{-1} = \begin{bmatrix} \frac{1}{21} & \frac{-1}{6} \\ \frac{-1}{6} & \frac{5}{6} \end{bmatrix}$$

$$M^T b = \begin{bmatrix} 1 & 2 & 4 & 7 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 30 \\ 27 \\ 21 \\ 14 \end{bmatrix} = \begin{bmatrix} 266 \\ 92 \end{bmatrix}$$

Then solve $M^T M z = M^T b$ by multiplication by inverse,

$$z = (M^T M)^{-1} M^T b = \begin{bmatrix} \frac{1}{21} & \frac{-1}{6} \\ \frac{-1}{6} & \frac{5}{6} \end{bmatrix} \begin{bmatrix} 266 \\ 92 \end{bmatrix} = \begin{bmatrix} -8/3 \\ 97/3 \end{bmatrix} = \begin{bmatrix} -2.67 \\ 32.33 \end{bmatrix}$$

Therefore, $\boxed{y = -2.67x + 32.33 = \frac{-8}{3}x + \frac{97}{3}}$

§6.4 #16 An inconsistent system is given below. Find the vector(s) z for which $\|Az - b\|$ is minimized. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 3 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}$

If $Ax = b$ has no sol^{n} then we argued in lecture that $A^T A z = A^T b$ yields sol^{n} z closest to solving the inconsistent system $Ax = b$. Consider then,

$$A^T A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 6 \\ 6 & 6 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 20 \\ 17 \end{bmatrix}$$

Notice $(A^T A)^{-1} = \begin{bmatrix} 1/5 & -1/5 \\ -1/5 & 11/30 \end{bmatrix}$ thus

$$z = (A^T A)^{-1} A^T b = \begin{bmatrix} 1/5 & -1/5 \\ -1/5 & 11/30 \end{bmatrix} \begin{bmatrix} 20 \\ 17 \end{bmatrix} = \boxed{\begin{bmatrix} 3/5 \\ 67/30 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 2.233 \end{bmatrix}}$$

§6.4 #38 A space vehicle is launched and $y = a + bt + ct^2$ models it's position y at time t. Find best fit for a, b, c given the following table of data,

t	5	10	15	20	25	30
y	140	290	560	910	1400	2000

Plug in the data:

$$\begin{aligned}
140 &= a + 5b + 25c \\
290 &= a + 10b + 100c \\
560 &= a + 15b + 225c \\
910 &= a + 20b + 400c \\
1400 &= a + 25b + 625c \\
2000 &= a + 30b + 900c
\end{aligned}$$

Convert to matrix problem:

$$\underbrace{\begin{bmatrix} 1 & 5 & 25 \\ 1 & 10 & 100 \\ 1 & 15 & 225 \\ 1 & 20 & 400 \\ 1 & 25 & 625 \\ 1 & 30 & 900 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 140 \\ 290 \\ 560 \\ 910 \\ 1400 \\ 2000 \end{bmatrix}}_{b}$$

We find best-fit sol$^{\underline{n}}$ by solving the corresponding normal-eq$^{\underline{n}}$s $M^TMz = M^Tb$ then $\|Mz - b\|$ is minimized (it's the best fit)
Using my handy-dandy TI-89,

$$M^TM = \begin{bmatrix} 6 & 105 & 2275 \\ 105 & 2275 & 55125 \\ 2275 & 55125 & 1421880 \end{bmatrix} \qquad M^Tb = \begin{bmatrix} 5300 \\ 125200 \\ 3197500 \end{bmatrix}$$

$$M^TMz = M^Tb \Rightarrow z = (M^TM)^{-1}M^Tb = \begin{bmatrix} 107 \\ -4.07857 \\ 2.23571 \end{bmatrix}$$

$$\therefore \boxed{y = 107 - 4.07857t + 2.23571t^2}$$

(ignoring sig-fig considerations)