This Mission intends to reinforce Lectures 2 - Lecture 6 of my 2022 Mathematics of GR course. Please read Chapter 1 of Sean Carrol's *Spacetime and Geometry* text.

**Problem 1** Use the repeated index notation to prove the identity $\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\bullet\vec{C})\vec{B} - (\vec{A}\bullet\vec{B})\vec{C}$

**Problem 2** Consider two inertial frames which are related by an $x$-velocity transformation. In particular, we suppose the primed frame is moving with velocity $V$ in the $x$-direction. We have the following relations between frame $(t, x, y, z)$ and $(t', x', y', z')$:

$$\begin{aligned} t' &= \gamma(t - Vx/c^2) \\ x' &= \gamma(x - Vt) \\ y' &= y, \\ z' &= z \end{aligned}$$

where $\gamma = \dfrac{1}{1 - V^2/c^2}$ and $c$ is the speed of light. Notice these are linear relations between $(t, x, y, z)$ and $(t', x', y', z')$ hence we have the corresponding differential relations:

$$\begin{aligned} dt' &= \gamma(dt - V dx/c^2) \\ dx' &= \gamma(dx - V dt) \\ dy' &= dy, \\ dz' &= dz \end{aligned}$$

Let $\vec{u} = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle$ denote the velocity of a trajectory in the $(t, x, y, z)$-frame and let $\vec{u}' = \langle \frac{dx'}{dt'}, \frac{dy'}{dt'}, \frac{dz'}{dt'} \rangle$ denote the velocity of the same trajectory as measured in the $(t', x', y', z')$ frame. Furthermore, let us denote $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{u}' = \langle u_1', u_2', u_3' \rangle$. Using formal calculus of differentials derive the following relations between the velocity measured in the unprimed and primed frame:

**(a.)** $u_1' = \dfrac{u_1 - V}{1 - Vu_1/c^2}$

**(b.)** $u_2' = \dfrac{u_2}{\gamma(1 - Vu_1/c^2)}$

**(c.)** $u_3' = \dfrac{u_3}{\gamma(1 - Vu_1/c^2)}$

**Problem 3** Use the transformations derived in the previous problem to answer the following:

**(a.)** Suppose $\vec{u} = \langle c, 0, 0 \rangle$. Show $\vec{u}'$ has speed $c$.

**(b.)** Suppose $\vec{u} = \langle 0, c, 0 \rangle$. Show $\vec{u}'$ has speed $c$.

**Problem 4** The speed of an object moving with $x$-velocity $u_1'$ in a primed frame moving with velocity $V$ in the $x$-direction is given by $u_1 = \dfrac{u_1' + V}{1 + Vu_1'/c^2}$ (this is the inverse of the relation in part (a.) of a previous problem). This is the colinear velocity addition rule in special relativity. As you can clearly see, *velocities do not add*. In contrast, if we define rapidity of the frame by

$$\tanh(\alpha) = V/c$$

and the rapidity of the particle in the primed frame by

$$\tanh(\beta') = u_1'/c$$

and the rapidity of the particle in the unprimed frame by

$$\tanh(\beta) = u_1/c$$

then show that $\beta = \alpha + \beta'$. That is, show that *rapidities add*. Let me give you a huge hint:

$$\tanh(a+b) = \frac{\tanh(a) + \tanh(b)}{1 + \tanh(a)\tanh(b)}.$$

**Problem 5** Answer the questions posed on page 6 of Lecture 4. Prove or disprove:

- **(a.)** The composition of isometries in Minkowksi space is an isometry.
- **(b.)** A spacetime translation is an isometry on $\mathbb{R}^4$.
- **(c.)** A Lorentz matrix induces an isometry on spacetime.
- **(d.)** An isometry of Minkowski space fixing 0 is a Lorentz transformation.
- **(e.)** Every isometry of spacetime is the composite of a Lorentz transformation and a spacetime translation.
- **(f.)** The Poincare group is a group.

**Problem 6** Show that a trajectory which falls along a line in one inertial frame of reference will also fall on a line in any other inertially related frame of reference. The context of this problem is Newtonian space.

**Problem 7** Derive Coriolis effect. See http://www.supermath.info/CoriolisEffect.pdf
Notice, this is somewhat removed from the formulas we derived in Lecture 5 page 2. I would like to understand how to get from one formalism to the other. In particular, the boxed equation on page 4 of the linked pdf which states:

$$\vec{a}_S = \vec{a}_0 + \vec{a}_{\bar{S}} + 2\vec{\omega} \times \vec{v}_{\bar{S}} + \frac{d\vec{\omega}}{dt} \times \vec{\bar{r}} + \vec{\omega} \times (\vec{\omega} \times \vec{\bar{r}})$$

versus my Lecture 5 page 2 formula of:

$$a_y = \ddot{R}x + 2\dot{R}v_x + Ra_x + \ddot{p}$$

**Problem 8** Carrol, Exercise 7 on page 47 on tensor calculations for given numerical tensor and 4-vector

**Problem 9** Carrol, Exercise 10 on page 47 on transforming electric and magnetic fields