This Mission intends to reinforce Lectures 6 - Lecture 10 of my 2022 Mathematics of GR course. Please read Chapter 1 of Sean Carrol's *Spacetime and Geometry* text.

Problem 10 Suppose $F_{\mu\nu}$ and $G^{\mu\nu}$ are tensors on spacetime. Show that $F_{\mu\nu}G^{\mu\nu}$ is an invariant. In particular, transform the expression to another frame of reference via a Lorentz transformation and show the value is unaltered. Recall,

$$F_{\mu'\nu'} = \Lambda^{\mu}_{\mu'}\Lambda^{\nu}_{\nu'}F_{\mu\nu} \qquad \& \qquad G^{\mu'\nu'} = \Lambda^{\mu'}_{\mu}\Lambda^{\nu'}_{\nu}G^{\mu\nu}$$

and $\Lambda^{\sigma}_{\alpha'}\Lambda^{\alpha'}_{\beta} = \delta^{\sigma}_{\beta}$ and $\Lambda^{\beta}_{\alpha'}\Lambda^{\sigma'}_{\beta} = \delta^{\sigma'}_{\alpha'}$. Notice the placement of the indices defines the matrix. We distinguish between $\Lambda^{\beta}_{\alpha'}$ and $\Lambda^{\alpha'}_{\beta}$, these are inverse of one another.

Problem 11 Work out the equations of motion for the simple pendulum of length L which hangs from a mount and makes angle θ with the vertical. You don't have to solve the equations of motion (I don't think they are solvable by elementary means, we usually just present the small angle solution for the introductory physics course)

Problem 12 Suppose a ball slides frictionlessly on a hemispherical bowl of radius R on the surface of earth. Assume the Earth's rotation around its axis and the Sun are neglible as are the gravitational forces from the Moon, Mars and the force of air resistance etc... Suppose $\theta = \pi/2$ for the rim of the bowl and $0 \leq \phi \leq 2\pi$ sweeps out the bowl. I'm advocating using Physics notation (see E5 in Lecture 9 for the formulas which place the origin at the center of the bowl). We should use Kinetic energy

$$T = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right)$$

The potential energy due to gravity can be written as $U = mgz = mgr\cos\theta$. Then use

$$L = T - U + \lambda(r - R)$$

where λ is a Lagrange-multiplier in the sense shown in E6 of Lecture 9. Find the equations of motion for the ball and the equation for the normal force on the ball as a function of the spherical coordinates based at the center of the bowl.

Problem 13 Suppose we use hyperbolic coordinates to model Euclidean space. In particular, let ρ, φ be coordinates for $\mathbb{R}^2$ which are related to Cartesian coordinates according to the relations:

$$x = \rho\cosh\varphi \qquad \& \qquad y = \rho\sinh\varphi$$

and we assume $\rho > 0$ thus the ρ, φ coordinates only serve to parametrize the sector of the plane bounded by $y = \pm x$ where $x > 0$. Find the equations of motion for a free particle in terms of the hyperbolic coordinates.

Problem 14 Consider two particles with 4-momenta P_1 and P_2 respectively. Suppose the particles collide **elastically**. Let P_1' and P_2' denote the 4-momenta of the particles after the collision. Conservation of momentum and energy are simultaneously combined in the conservation of 4-momentum. Assume 4-momentum of the system is conserved and prove $P_1 \bullet P_2 = P_1' \bullet P_2'$ where we intend the $\bullet$ to denote the Minkowski product:

$$V \bullet W = -V^1W^1 + V^1W^1 + V^2W^2 + V^3W^3$$

Problem 15 (McComb's *Dynamics and Relativity*, page 249-250) Two particles of rest mass m_1 and m_2 have velocities $\vec{u_1}$ and $\vec{u_2}$ as measured in a given inertial reference frame. Suppose these coalesce into a single particle of mass M and velocity $\vec{u}$.

(a.) use conservation of 4-momentum to deduce that

$$\vec{u} = \frac{m_1\gamma_1\vec{u_1} + m_2\gamma_2\vec{u_2}}{m_1\gamma_1 + m_2\gamma_2}$$

where $\gamma_1 = \gamma(u_1)$ and $\gamma_2 = \gamma(u_2)$.

(b.) What is the minimum mass M_{min} of a particle which can decay into two other particles of mass m_1 and m_2 ?

Problem 16 The next problem counts double it's taken from McComb's text on pages 254-255. He also gives a solution without using invariance of the Minkowski product, but I'm trying to showcase how invariance allows us to solve interesting problems in Physics.

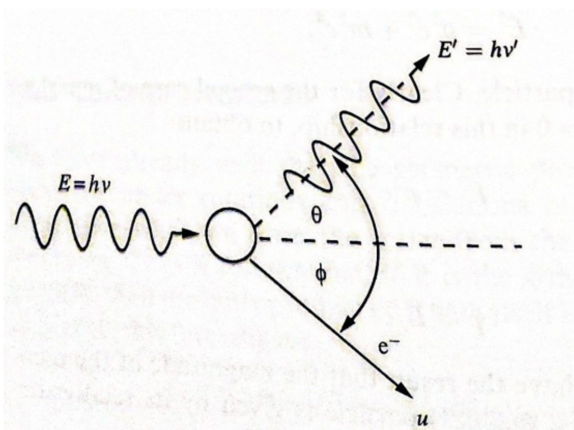
Problem 17 Massless particles link energy and momentum according to the relation $E = pc$. The 4-momentum of a massless particle travelling in the direction $\hat{n}$ is given by $Q = \left\langle \frac{E}{c}, \frac{E}{c}\hat{n} \right\rangle$. Furthermore, the wave-particle duality relates the energy of a massless particle to its frequency according to Planck's constant; $E = hf$ or wavelength $\lambda = c/f$. So, we can restate the 4-momentum for a photon as

$$Q = \frac{hf}{c}\langle 1, \hat{n} \rangle .$$

Suppose a photon with momentum $p = h/\lambda = hf/c$ and energy hf, collides with a stationary electron of mass m, and is scattered at an angle θ, with new energy hf'. Show that the change of energy of the photon is related to the scattering angle by

$$\lambda' - \lambda = 2\lambda_c \sin^2 \theta/2$$

where $\lambda_c = \frac{h}{mc}$ is known as the Compton wavelength. The diagram below uses $f = \nu$ to denote frequency. This is a picture depicting the collision as measured in the LAB-frame.



Let me walk you through how to derive this result via 4-momentum based arguments:

(a.) Let P_1 and P_1' denote the 4-momentum of the photon before and after the scattering. Let P_2 and P_2' denote the 4-momentum before and after the scattering respectively. Write $P_1 = \frac{hf}{c}\langle 1, \hat{n}\rangle$ and $P_1' = \frac{hf'}{c}\langle 1, \hat{n}'\rangle$. Show $P_1 \bullet P_1 = 0$ and $P_1' \bullet P_1' = 0$.

(b.) Express $P_2 = \langle mc, 0\rangle$ since the electron is at rest in the LAB-frame before the scattering and $P_2' = \langle m\gamma' c, \vec{p}'\rangle$ where $\gamma' = \gamma(u')$ and $\vec{p}'$ is the relativistic momentum of the recoiling electron after the scattering. Show $P_2 \bullet P_2 = P_2' \bullet P_2'$.

(c.) Conservation of 4-momentum of the system gives $P_1 + P_2 = P_1' + P_2'$. Solve for P_2' and observe

$$P_2' \bullet P_2' = (P_1 + P_2 - P_1') \bullet (P_1 + P_2 - P_1')$$

use what you already have shown in parts (a.) and (b.) to simplify the above and derive that:

$$P_1 \bullet P_2 - P_2 \bullet P_1' = P_1 \bullet P_1'$$

(d.) Derive from the result above

$$mhf - mhf' = \frac{h^2 f f'}{c^2}(1 - \hat{n} \bullet \hat{n}')$$

(e.) Finally, note $\hat{n} \bullet \hat{n}' = \cos\theta$ to show $\lambda' - \lambda = 2\lambda_c \sin^2\theta/2$

The next problem is taken from McComb's text on pages 256-257. The problem illustrates how to use appropriate reference frames to simplify the analysis of a collision in relativistic dynamics. The center of momentum frame is often useful in such problems.

Problem 18 A photon of energy E is asorbed by a stationary proton, resulting in a neutral pion and a recoiling proton. Show the threshold energy for a photon to cause this process is:

$$E_{min} = \frac{m(m + 2M)c^2}{2M}$$

where m is the pion mass and M is the proton mass. I'll walk you through how to show this result using a frames of reference technique:

(a.) In the Center of Momentum Frame (CM) we have total 3-momentum is zero by definition of the frame. The minimum energy occurs when the after collision particles have zero velocity which give γ-factors of 1 hence $E_{CM} = mc^2 + Mc^2 = (m + M)c^2$. Then the total 4-momentum in the CM-frame at the threshold energy is:

$$P_{CM} = \left\langle \frac{E_{CM}}{c}, \sum \vec{p} \right\rangle = \left\langle \frac{E_{CM}}{c}, 0 \right\rangle$$

Calculate $P_{CM} \bullet P_{CM}$.

(b.) In the Laboratory or target frame (LAB) the proton is at rest thus $P_{proton} = \langle Mc, 0\rangle$ and the photon has $P_{photon} = (E/c, \vec{p})$ where $\|\vec{p}\| = E/c$. Let $P_{LAB} = P_{proton} + P_{photon}$ be the total 4-momentum in the LAB frame. Calculate P_{LAB} then compute $P_{LAB} \bullet P_{LAB}$.

(c.) Use $P_{CM} \bullet P_{CM} = P_{LAB} \bullet P_{LAB}$ to derived the threshold energy formula.

Problem 19 Suppose the Lagrangian for electromagnetism given in Lecture 10 is modified to include an extra term which couples A_μ to itself:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu + mA_\mu A^\mu$$

Find how Maxwell's equations are modified by the additional term proportional to m.