PRINT NAME: ______________________________.

MATH 423: HOMEWORK: SUBGROUPS AND ISOMORPHISM ASSIGNMENT

Show **your** work carefully. Use full sentences, proper grammar and be precise. You don't have to copy the problem statement again, but, your solution must be self-contained. 60pts to earn here.

Problem 13: For each of the following subsets of $\mathbb{R}^{2\times 2}$ determine whch (if any) of the subsets below form a subgroup of $\mathrm{GL}_2(\mathbb{R})$,

(a.) $A = \left\{ \begin{bmatrix} a & 1 \\ 0 & b \end{bmatrix} \;\middle|\; ab \neq 0 \right\}$,

(b.) $B = \left\{ \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} \;\middle|\; bc \neq 0 \right\}$,

(c.) $C = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix} \;\middle|\; c \neq 0 \right\}$.

Problem 14: Recall $C(a) = \{x \in G \mid xa = ax\}$ is the **centralizer of** a and $Z(G) = \{x \in G \mid xg = gx \text{ for each } g \in G\}$ is the **center of** G. Calculate the following:

(a.) Let $G = \mathbb{Z}_n$ under addition. Calculate $C(a)$ for each $a \in G$ and find $Z(G)$

(b.) Let $G = \mathrm{GL}_2(\mathbb{R})$ if $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ find $C(A)$.

Remark: *as you ponder (a.) it would be a good time to remember the concept of a group allows different kinds of operations for various sets, however we tend to use multiplicative notation to define various concepts. It is sometimes necessary to translate an idea from the world of multiplicative notation to the word of additive or other notation. For example, ab is replaced with $a + b$ in $G = (S, +)$ or $a \star b$ in $G = (S, \star)$. It's a good exercise to translate $Z(G)$ or $C(a)$ into the $+$ or $\star$ operations.*

Problem 15: Consider $U(12) = 1, 5, 7, 11$. Find the order of each element in $U(12)$.

Problem 16: Recall the set of bijections on $\mathbb{R}^n$ forms a group under function composition and we denote it by $\mathrm{Perm}(\mathbb{R}^n)$. The Euclidean distance between x and y is given by $d(x, y) = \sqrt{(y - x) \bullet (y - x)}$. Let $\mathrm{Isom}(\mathbb{R}^n)$ denote the set of Euclidean distance-preserving bijections of $\mathbb{R}^n$; more precisely, $\sigma \in \mathrm{Isom}(\mathbb{R}^n)$ if σ is a bijection on $\mathbb{R}^n$ and $d(\sigma(x), \sigma(y)) = d(x, y)$ for all $x, y \in \mathbb{R}^n$.

(a.) Show $\mathrm{Isom}(\mathbb{R}^n) \leq \mathrm{Perm}(\mathbb{R}^n)$ by the subgroup test,

(b.) Let $H = \{t_b \mid t_b(x) = x + b \text{ for all } x \in \mathbb{R}^n\}$. Show $H \leq \mathrm{Isom}(\mathbb{R}^n)$. We usually call H the group of **translations** in $\mathbb{R}^n$,

Problem 17: Let $S = \left\{ \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} \;\middle|\; x, y \in \mathbb{Z}_3 \right\}$ and define $H = S \cap GL_2(\mathbb{Z}_3)$. Prove $H \leq GL_2(\mathbb{Z}_3)$ and find an isomorphism from H to S_3.

Remark: *Notice $GL_2(\mathbb{Z}_3)$ is the set of invertible 2×2 matrices with entries from $\mathbb{Z}_3$, we explained how to judge invertibility of a 2×2 matrix over $\mathbb{Z}_n$ in Problem 5 of this course. Also, please note an isomorphism is a function, so you need to explain how each element in H maps to a specific permutation in $S_3 = \{(1), (12), (13), (23), (123), (231)\}$.*

Problem 18: Let G be a group. Define $\Psi : G \to G$ by $\Psi(x) = x^{-1}$ for each $x \in G$.

(a.) Prove Ψ is injective and surjective.

(b.) Prove Ψ is an isomorphism if and only if G is abelian.

Problem 19: Chapter 3, Exercise # 56 (group problem)

Problem 20: Chapter 3, Exercise # 67 (group problem)

Problem 21: Chapter 3, Exercise # 71 (group problem) and find an isomorphism from $\mathbb{Z}_4$ to a subgroup of H.

Problem 22: Chapter 6, Exercise # 4 (groups not isomorphic)

Problem 23: Chapter 6, Exercise # 31 (working with isomorphisms)

Problem 24: Prove that the group of nonzero real numbers under multiplication is not isomorphic to the group of nonzero complex numbers under multiplication.

Remark: *Theorem 2.4.25 on page 55 of my 2018 abstract algebra notes is helpful here.*