The following problems are adapted from your text. These are recommended homework.

The main distinction between my lecture and the text is that I use the imaginary exponential notation $e^{i\theta} = \cos\theta + i\sin\theta$ rather than the clumsy notation $cis(\theta) = \cos\theta + i\sin\theta$.

Problem 1: Find the length (also known as absolute value) of the following complex numbers:

(a.) $5 + 3i$,
(b.) $-7 + i$,
(c.) $-3 - 3i$,
(d.) $\sqrt{2} - 6i$,
(e.) $2i$.

Problem 2: Write the following complex numbers in polar form. That is, find $r \geq 0$ and θ such that $z = re^{i\theta}$ for

(a.) $z = 2 + 2i$,
(b.) $z = 8 - 4i$,
(c.) $z = -\frac{1}{2} - \frac{1}{2}i$,
(d.) $z = \sqrt{3} + i$,
(e.) $z = 3i$.

Problem 3: Give z in polar form as below (note, I write $e^{i\theta} = \exp(i\theta)$ to make it easier to read), find $x, y \in \mathbb{R}$ for which $z = x + iy$. That is, find the Cartesian form of each z given below. Also graph the number in the complex plane to check your answer.

(a.) $z = 7\exp(i\pi/6)$,
(b.) $z = 2\exp(i\pi/3)$,
(c.) $z = 4\exp(i7\pi/6)$,

Problem 4: Another notation which is helpful to express polar coordinates of a complex number is the following: if $z = r\cos\theta + ir\sin\theta$ with $r = |z| \geq 0$ then $\angle(z) = \theta$. Find the Cartesian form of z given that:

(a.) $|z| = 7$, $\angle(z) = 25^o$,
(b.) $|z| = 3$, $\angle(z) = 240^o$,
(c.) $|z| = \sqrt{2}$, $\angle(z) = 100^o$,

Problem 5: Given z_1, z_2 as below find the polar form of their product $z_1 z_2$

(a.) $z_1 = 13e^{i\pi/4}$, $z_2 = 5e^{-i\pi/3}$,
(b.) $z_1 = 10e^{i\pi/7}$, $z_2 = -5$,
(c.) $z_1 = 3e^{i2\pi/3}$, $z_2 = \frac{1}{4}e^{i\pi/3}$,
(d.) $z_1 = \sqrt{5}e^{i5\pi/8}$, $z_2 = \sqrt{15}e^{i\pi/12}$,
(e.) $z_1 = 1 + i$, $z_2 = 3 - 4i$,

(f.) $z_1 = 2+i$, $z_2 = 2+i$.

Problem 6: Given z_1, z_2 as below find the polar form of their quotient $\frac{z_1}{z_2}$

(a.) $z_1 = 13e^{i\pi/4}$, $z_2 = 5e^{-i\pi/3}$,

(b.) $z_1 = 10e^{i\pi/7}$, $z_2 = -5$,

(c.) $z_1 = 3e^{i2\pi/3}$, $z_2 = \frac{1}{4}e^{i\pi/3}$,

(d.) $z_1 = \sqrt{5}e^{i5\pi/8}$, $z_2 = \sqrt{15}e^{i\pi/12}$,

(e.) $z_1 = 1+i$, $z_2 = 3-4i$,

(f.) $z_1 = 2+i$, $z_2 = 2+i$.

Problem 7: Find z^n in polar form for the following,

(a.) $z = 7\exp(i\pi/6)$, find polar form of z^6

(b.) $z = 2\exp(i\pi/3)$, find polar form of z^2

(c.) $z = 4\exp(i7\pi/6)$, find polar form of z^3

(d.) $|z| = 7$, $\angle(z) = 25^o$, find polar form of z^6

(e.) $|z| = 3$, $\angle(z) = 240^o$, find polar form of z^3

Problem 8: Given $z = re^{i\theta}$ where $r = |z|$ we define the **principle** n-th root of z by

$$\sqrt[n]{z} = \sqrt[n]{r}\exp(i\theta/n).$$

In contrast, the **set of n-th roots** of z are

$$z^{1/n} = \{\sqrt[n]{z},\ \sqrt[n]{z}\zeta_n,\ \sqrt[n]{z}\zeta_n^2, \dots,\ \sqrt[n]{z}(\zeta_n)^{n-1}\}$$

where $\zeta_n = \exp(2\pi i/n)$ is the **principle n-th root of unity**. Calculate $\sqrt[n]{z}$ and $z^{1/n}$ for the following complex numbers and choices of n,

(a.) $z = 16\exp(4\pi i/3)$ and $n = 2$,

(b.) $z = -16$ and $n = 2$,

(c.) $z = 27\exp(2\pi i/3)$ and $n = 3$,

(d.) $z = 8\exp(7\pi i/4)$ and $n = 3$,

(e.) $z = 16\exp(2\pi i/3)$ and $n = 4$,

Problem 9: We derived in lecture that $\cos\theta = \dfrac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$ and $\sin\theta = \dfrac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$. We also showed the imaginary exponential satisfies $e^{i\alpha}e^{i\beta} = e^{i(\alpha+\beta)}$. Derive the following trigonometric identities via an algebraic arguments using imaginary exponentials:

(a.) $\cos^2\theta = \dfrac{1}{2}\left(1 + \cos(2\theta)\right)$ *(I did this one in lecture.)*

(b.) $\cos^3(\theta) = \dfrac{3}{4}\cos(\theta) + \dfrac{1}{4}\cos(\theta)$

(c.) $\cos^4(\theta) = \dfrac{3}{8} + \dfrac{1}{2}\cos(2\theta) + \dfrac{1}{8}\cos(4\theta)$

(d.) $\cos(2x)\sin(3x) = \dfrac{1}{2}\sin(x) + \dfrac{1}{2}\sin(5x)$

Problem 10: (bonus) The distance between z_1 and z_2 is the length of the complex number $z_2 - z_1$; if we denote the distance from z_1 to z_2 by $d(z_1, z_2)$ then $d(z_1, z_2) = |z_2 - z_1|$. Certain curves in the xy-plane can be described in terms of distance. In such a case, we can use complex notation to describe the curve.

(a.) a circle with radius R centered at (h, k) can be described as the collection of points distance R from the center (h, k). In complex notation, if $z = x + iy$ satisfies $|z - (h + ik)| = R$ then z is on the circle. Recall $|w|^2 = w\overline{w}$ and show that the equation $|z - (h+ik)|^2 = R^2$ is equivalent to the standard equation of the circle with radius R centered at (h, k).

(b.) Let $z_1 = a + ib$ and $z_2 = c + id$. Let C be the curve given by all $z = x + iy$ for which $|z - z_1| = |z - z_2|$. Find the Cartesian equation of this curve. Name the curve.

(c.) Let C be the curve given by all $z = x + iy$ for which $|z + i| - |z - i| = 1$. Find the Cartesian equation of this curve. Name the curve.

(d.) Let C be the curve given by all $z = x + iy$ for which $|z + i| + |z - i| = 1$. Find the Cartesian equation of this curve. Name the curve.

(e.) Let C be the curve given by all $z = x + iy$ for which $|z + 3| + |z - 3| = 1$. Find the Cartesian equation of this curve. Name the curve.