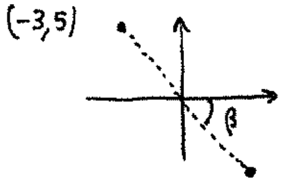
NAME ______________________________

MATH 114: FALL 2021	TAKEHOME QUIZ 5 (20PTS)

2pts per problem. Show your work.

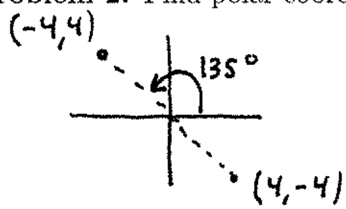
Problem 1: Find polar coordinates for the point $(-3, 5)$. Use degrees for the angle please.



$$\beta = \tan^{-1}\left(\frac{5}{-3}\right) \implies \theta = 180° + \tan^{-1}\left(\frac{5}{-3}\right) = \boxed{120.96°}$$

$$r = \sqrt{(-3)^2 + 5^2} = \boxed{\sqrt{34}}$$

Problem 2: Find polar coordinates r, θ for the point $(4, -4)$ such that $r < 0$. Use degrees for θ.



$$r = -\sqrt{(4)^2 + (-4)^2} \cong \boxed{-5.657}$$

$$\boxed{\theta = 135°}$$

Problem 3: Given $z = 3\sqrt{2}e^{i\pi/4}$ and $w = 2i$ calculate the Cartesian form and polar form of zw.

$$i = e^{\pi i/2} = \cos(\pi/2) + i\sin(\pi/2) = i$$

$$zw = (3\sqrt{2}\, e^{i\pi/4} \cdot 2 e^{i\pi/2})$$

$$= 6\sqrt{2}\exp\left(i\left(\frac{\pi}{4} + \frac{\pi}{2}\right)\right)$$

$$= 6\sqrt{2}\, e^{3\pi i/4}$$

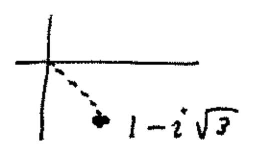
$$\underbrace{\boxed{r = 6\sqrt{2}} \text{ and } \boxed{\theta = \frac{3\pi}{4}}}_{\text{polar coord. for } zw}$$

$$zw = 6\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$$

$$= 6\sqrt{2}\left(\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

$$= \underbrace{\boxed{-6 + 6i}}_{\text{Cartesian form of } zw}$$

Problem 4: Let $z = 1 - i\sqrt{3}$. Calculate the Cartesian and polar form of z^5.



$$|z| = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$$

$$\theta = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right) = \tan^{-1}\left(\frac{\sqrt{3}/2}{-1/2}\right) = \tan^{-1}(\tan(-60°)) = -60°$$

$$\theta = \frac{-2\pi}{6} = \frac{-\pi}{3}$$

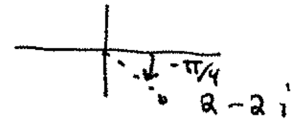
$$z = 2e^{\frac{-\pi i}{3}}$$

polar form

$$z^5 = \left(2e^{\frac{-\pi i}{3}}\right)^5 = \boxed{32\, e^{\frac{-5\pi i}{3}}} = 32\left(\cos\left(\frac{-5\pi}{3}\right) + i\sin\left(\frac{-5\pi}{3}\right)\right)$$

$$= 32\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

$$= \underbrace{\boxed{16 - i(16\sqrt{3})}}_{\text{Cartesian Form}}$$



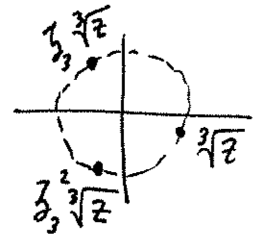
Problem 5: Calculate $\sqrt[3]{2-2i}$ and the set of values $(2-2i)^{1/3}$

$$z = 2-2i = \sqrt{2^2+(-2)^2}\,\exp\left(\frac{-i\pi}{4}\right) = \sqrt{8}\,e^{-\frac{\pi i}{4}}$$

$$\sqrt[3]{z} = \sqrt[3]{\sqrt{8}}\;e^{\frac{-\pi i}{3\cdot 4}} = \boxed{\sqrt[6]{8}\,e^{\frac{-\pi i}{12}}}$$

$$z^{1/3} = \{\sqrt[3]{z},\ \zeta_3\sqrt[3]{z},\ \zeta_3^2\sqrt[3]{z}\} \quad \text{where } \zeta_3 = e^{\frac{2\pi i}{3}} = e^{\frac{8\pi i}{12}}$$

$$\Rightarrow \boxed{z^{1/3} = \left\{\sqrt[6]{8}\,e^{\frac{-\pi i}{12}},\ \sqrt[6]{8}\,e^{\frac{7\pi i}{12}},\ \sqrt[6]{8}\,e^{\frac{15\pi i}{12}}\right\}}$$



Problem 6: Factor $z^3 - 2 + 2i$ completely over $\mathbb{C}$.

$z^3 = 2-2i$ has sol'n's given in $z^{1/3}$

If $f(z) = z^3 - 2 + 2i$ then $f(c) = 0$ for each $c \in z^{1/3}$

thus by factor Th$^{\underline{m}}$,

$$z^3 - 2 + 2i = \boxed{\left(z - \sqrt[6]{8}\,e^{\frac{-\pi i}{12}}\right)\left(z - \sqrt[6]{8}\,e^{\frac{7\pi i}{12}}\right)\left(z - \sqrt[6]{8}\,e^{\frac{15\pi i}{12}}\right)}$$

Problem 7: Derive $\cos(2x)\sin(3x) = \frac{1}{2}\sin(x) + \frac{1}{2}\sin(5x)$ by using the algebra of the imaginary exponentials and the identities $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \frac{1}{2}(e^{i\theta} - e^{-i\theta})$

$$\begin{aligned}
\cos(2x)\sin(3x) &= \frac{1}{2}\left(e^{2ix} + e^{-2ix}\right)\frac{1}{2i}\left(e^{3ix} - e^{-3ix}\right) \\
&= \frac{1}{4i}\left(e^{5ix} - e^{-5ix} + e^{ix} - e^{-ix}\right) \\
&= \frac{1}{2}\left[\frac{1}{2i}\left(e^{5ix} - e^{-5ix}\right)\right] + \frac{1}{2}\left[\frac{1}{2i}\left(e^{ix} - e^{-ix}\right)\right] \\
&= \boxed{\frac{1}{2}\sin(5x) + \frac{1}{2}\sin(x)}
\end{aligned}$$

Key Concept:
$$\begin{aligned}
e^{i\theta} &= \cos\theta + i\sin\theta \\
\pm\quad e^{-i\theta} &= \cos\theta - i\sin\theta \\
\hline
e^{i\theta} + e^{-i\theta} &= 2\cos\theta \\
e^{i\theta} - e^{-i\theta} &= 2i\sin\theta
\end{aligned}$$

$\therefore \cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$

$\therefore \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$

can covert sine & cosine to imaginary exponentials which have $e^{i\theta}e^{i\beta} = e^{i(\theta+\beta)}$

$x = r\cos\theta$
$y = r\sin\theta$
$x^2+y^2=r^2$

Problem 8: Find the Cartesian form of the polar equation $r^2 = 4r\sin\theta$.

$$\boxed{x^2+y^2=4y}$$

$$x^2+y^2-4y=0$$
$$x^2+(y-2)^2=2^2$$

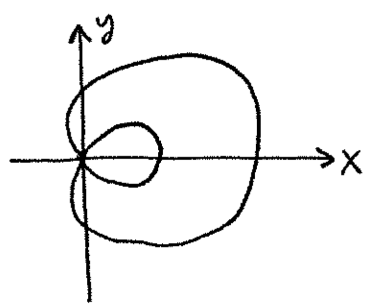
(it's a circle centered at $(0,2)$ with radius 2)

Problem 9: Find the polar form of the equation $y = 2x+3$. Please solve for r as a function of θ.

$$y=2x+3 \Rightarrow r\sin\theta = 2r\cos\theta+3$$
$$\Rightarrow r\sin\theta - 2r\cos\theta = 3$$
$$\Rightarrow r(\sin\theta-2\cos\theta)=3$$
$$\therefore \boxed{r=\frac{3}{\sin\theta-2\cos\theta}}$$

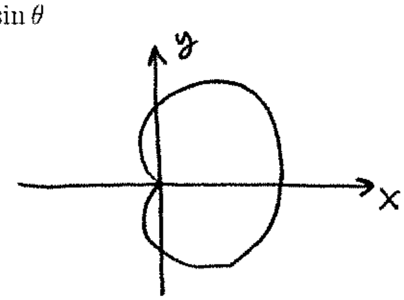
Problem 10: Classify the following polar equations, use the textbook and/or Desmos to find the graph and make a sketch of the shape. Name each curve.

(a.) $r = 1+2\cos\theta$



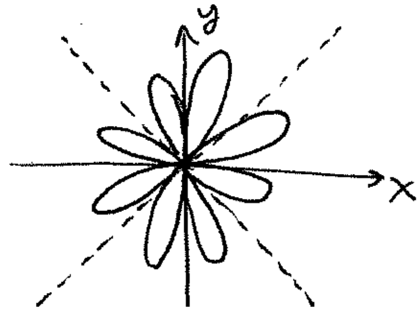
limacon
with inner loop

(b.) $r = 3+3\sin\theta$



Cardioid.

(c.) $r = \sin(4\theta)$



8 - petal flower