The text for this course is *Mathematical Analysis I* second edition by Beatriz Laferriere, Gerardo Laferriere and Nguyen Mau Nam. The exercises below are from this text. This homework covers the material discussed in Lectures 9,10,11,12 and 13. It is due 10-19-20.

Problem 41: Let c be a fixed real number. Define $a_n = c$ for all $n \in \mathbb{N}$. Prove $\lim_{n\to\infty}(a_n) = c$.

Problem 42: Exercise 2.1.1 part c.

Problem 43: Exercise 2.1.6 part a.

Problem 44: Exercise 2.1.10

Problem 45: Exercise 2.1.11

Problem 46: Exercise 2.1.12 part a.

Problem 47: Exercise 2.2.1

Problem 48: Exercise 2.2.2

Problem 49: Exercise 2.3.1

Problem 50: Exercise 2.3.3 part a.

Problem 51: Exercise 2.3.4

Problem 52: Exercise 2.3.6 part a.

Problem 53: Exercise 2.4.1

Problem 54: Exercise 2.4.2

Problem 55: Exercise 2.4.4

Problem 56: Exercise 2.4.5

Problem 57: Exercise 2.5.1 parts a and d

Problem 58: Exercise 2.5.2 part a

Problem 59: Exercise 2.5.4.

Problem 60: Exercise 2.2.5

Mission 3 Solution

$\boxed{P41}$ Let $a_n = c$ $\forall n \in \mathbb{N}$. Let $\varepsilon > 0$ and choose $N = 1$ then for $n \geq N$ observe $|a_n - c| = |c - c| = 0 < \varepsilon$. Thus $\lim_{n\to\infty}(a_n) = c$.

$\boxed{P42}$ Ex. 2.1.1 part c

Let $\varepsilon > 0$ then by ~~Archimedian~~ Principle $\exists N \in \mathbb{N}$ for which $\frac{1}{2}\left(1 + \frac{1}{\sqrt{\varepsilon}}\right) < N$. Suppose $n \geq N$ then notice

$$\frac{1}{2}\left(1+\frac{1}{\sqrt{\varepsilon}}\right) < n \Rightarrow \frac{1}{\sqrt{\varepsilon}} < 2n-1 \Rightarrow \frac{1}{\varepsilon} < (2n-1)^2$$

and thus $\frac{1}{(2n-1)^2} < \varepsilon$. Consider that,

$$\left|\frac{2n^3+1}{4n^3-n} - \frac{1}{2}\right| = \left|\frac{(2n^3+1)2 - (4n^3-n)}{2(4n^3-n)}\right|$$

$$= \left|\frac{4n^3+2-4n^3+n}{2n(4n^2-1)}\right|$$

$$= \left|\frac{n+2}{2n(2n+1)(2n-1)}\right|$$

$$< \frac{n+2}{(2n-1)(2n-1)(n+2)}$$

notice: $2n+1 \geq n+2$

$$= \frac{1}{(2n-1)^2}$$

$$< \varepsilon.$$

Therefore, $\lim_{n\to\infty}\left[\frac{2n^3+1}{4n^3-n}\right] = \frac{1}{2}$.

[P43] Ex. 2.1.6 part a.: Show $\lim_{n\to\infty}\left(\frac{n+\cos(n^2-3)}{2n^2+1}\right)=0$

Consider $-1<\cos(n^2-3)<1$ implies $n-1<n+\cos(n^2-3)<n+1$

thus $\frac{n-1}{2n^2+1}<\frac{n+\cos(n^2-3)}{2n^2+1}<\frac{n+1}{2n^2+1}$. However,

$\lim_{n\to\infty}\left(\frac{n-1}{2n^2+1}\right)=\lim_{n\to\infty}\left(\frac{1-1/n}{2n+1/n}\right)=0$, and $\lim_{n\to\infty}\left(\frac{n+1}{2n^2+1}\right)=\lim_{n\to\infty}\left(\frac{1+1/n}{2n+1/n}\right)=0.$

By Squeeze Th^{m} we find $\lim_{n\to\infty}\left[\frac{n+\cos(n^2-3)}{2n^2+1}\right]=0.$

[P44] Ex. 2.1.10

Suppose $a_n\geq 0$ $\forall n\in\mathbb{N}$ and $\lim_{n\to\infty}(a_n)=l$. Notice $\lim_{n\to\infty}(a_n)\geq\lim_{n\to\infty}(0)=0$ thus $l\geq 0$. If $l=0$ then suppose $\varepsilon>0$ and notice as $a_n\to 0$ we may select $N\in\mathbb{N}$ for which $n\geq N$ implies $|a_n|<\varepsilon^2$. Hence, $n\geq N$ implies $(\sqrt{a_n})^2<\varepsilon^2$ thus $(\sqrt{a_n})^2-\varepsilon^2<0$ or $(\sqrt{a_n}-\varepsilon)(\sqrt{a_n}+\varepsilon)<0$ and as $\sqrt{a_n}+\varepsilon>0$ we find $\sqrt{a_n}-\varepsilon<0$ thus $|\sqrt{a_n}|<\varepsilon$ and so $\sqrt{a_n}\to 0$ as $n\to\infty$.

Next, suppose $l>0$. Let $\varepsilon>0$ and choose $\bar{N}\in\mathbb{N}$ for which $n\geq\bar{N}$ implies $|a_n-l|<\varepsilon\sqrt{l}$. We can select such $\bar{N}$ as $a_n\to l$ and $\varepsilon\sqrt{l}>0$. Consider then, for $n\geq N$,

$$
\begin{aligned}
|\sqrt{a_n}-\sqrt{l}| &= \left|\frac{(\sqrt{a_n}-\sqrt{l})(\sqrt{a_n}+\sqrt{l})}{\sqrt{a_n}+\sqrt{l}}\right| \\
&= \left|\frac{a_n-l}{\sqrt{a_n}+\sqrt{l}}\right| \quad \text{decreased the denominator.} \\
&< \frac{|a_n-l|}{\sqrt{l}} \\
&< \frac{1}{\sqrt{l}}\varepsilon\sqrt{l} = \varepsilon \quad \therefore \lim_{n\to\infty}(\sqrt{a_n})=\sqrt{l} \quad //
\end{aligned}
$$

$\boxed{P45}$ Ex. 2.1.11 / Prove $\{a_n\}$ with $a_n = \sin\left(\frac{n\pi}{2}\right)$ is divergent.

Observe $a_{2k} = \sin\left(\frac{2k\pi}{2}\right) = \sin(k\pi) = 0$ thus $a_{2k} \rightarrow 0$.

However, $a_{4k+1} = \sin\left(\frac{(4k+1)\pi}{2}\right) = \sin\left(2k\pi + \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$

thus $a_{4k+1} \longrightarrow 1$ as $k \to \infty$. Thus $\{a_n\}$ is divergent since $Th^{\underline{m}}$ 2.1.9 implies all subsequences of a convergent sequence must likewise converge to the same value. Yet, we found two subsequences which converge to differing values.

$\boxed{P46}$ Ex. 2.1.12 part a / $\lim\limits_{n\to\infty} a_n = l \iff \lim\limits_{k\to\infty} a_{2k} = l$ and $\lim\limits_{k\to\infty} a_{2k+1} = l$.

$\Rightarrow$] Suppose $a_n \to l$. Let $\varepsilon > 0$ then $\exists N \in \mathbb{N}$ s.t. $n \geq N$ implies $|a_n - l| < \varepsilon$. Let $M = N/2$ and suppose $k \geq M$ then $k \geq N/2$ gives $2k \geq N$ hence $|a_{2k} - l| < \varepsilon \therefore a_{2k} \to l$. Likewise, let $\bar{M} = \frac{N-1}{2}$ and note $k \geq \bar{M}$ provides $k \geq \frac{N-1}{2}$ thus $2k \geq N-1$ or $2k+1 \geq N$ thus $|a_{2k+1} - l| < \varepsilon$ and consequently, $a_{2k+1} \longrightarrow l$ and this completes the $\Rightarrow$] part of the proof.

$\Leftarrow$] Suppose $\lim\limits_{k\to\infty}(a_{2k}) = l$ and $\lim\limits_{k\to\infty}(a_{2k+1}) = l$. Let $\varepsilon > 0$ and notice $\exists N_0 \in \mathbb{N}$ s.t. $k \geq N_0$ implies $|a_{2k} - l| < \varepsilon$. Likewise, $\exists N_1 \in \mathbb{N}$ s.t. $k \geq N_1$ implies $|a_{2k+1} - l| < \varepsilon$ as $a_{2k+1} \to l$. Let $N = \max\{2N_0, 2N_1+1\}$. Suppose $n \geq N$ then if $n = 2k$ we have $2k \geq 2N_0 \Rightarrow k \geq N_0 \Rightarrow |a_n - l| = |a_{2k} - l| < \varepsilon$. Likewise, if $n = 2k+1$ then $2k+1 \geq 2N_1+1$ hence $k \geq N_1$ and thus $|a_n - l| = |a_{2k+1} - l| < \varepsilon$. Thus $\forall n \geq N$ we find $|a_n - l| < \varepsilon$ and it follows $a_n \to l$. //

$\boxed{P47}$ Ex 2.2.1

(a.) $\lim_{n\to\infty}\left(\frac{3n^2-6n+7}{4n^2-3}\right) = \lim_{n\to\infty}\left(\frac{3-6/n+7/n^2}{4-3/n^2}\right) = \frac{3-0+0}{4-0} = \boxed{\frac{3}{4}}$

(b.) $\lim_{n\to\infty}\left(\frac{1+3n-n^3}{3n^3-2n^2+1}\right) = \lim_{n\to\infty}\left(\frac{1/n^3+3/n^2-1}{3-2/n+1/n^3}\right) = \frac{0+0-1}{3-0+0} = \boxed{\frac{-1}{3}}$

$\boxed{P48}$ Ex 2.2.2

(a.) $\lim_{n\to\infty}\left(\frac{\sqrt{3n}+1}{\sqrt{n}+\sqrt{3}}\right) = \lim_{n\to\infty}\left(\frac{\sqrt{3}+1/\sqrt{n}}{1+\sqrt{3}/\sqrt{n}}\right) = \frac{\sqrt{3}+0}{1+0} = \boxed{\sqrt{3}}$

(b.) $\lim_{n\to\infty}\sqrt[n]{\frac{2n+1}{n}} = \lim_{n\to\infty}\sqrt[n]{2+\frac{1}{n}}$

Observe $2 < 2+\frac{1}{n} \leq 3$ for all $n \in \mathbb{N}$

thus $\sqrt[n]{2} < \sqrt[n]{2+\frac{1}{n}} \leq \sqrt[n]{3}$ but by Example 2.2.2 in your text (pg. 39–40) note $\sqrt[n]{2}, \sqrt[n]{3} \to 1$ thus by Squeeze Thm $\lim_{n\to\infty}\sqrt[n]{2+\frac{1}{n}} = \lim_{n\to\infty}\sqrt[n]{\frac{2n+1}{n}} = \mathbf{1}.$

$\boxed{P49}$ Ex 2.3.1

Let $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{a_n + 2}$ for $n \geq 1$.

(a.) $a_1 = \sqrt{2} < 2$ thus $a_n < 2$ for $n = 1$. Suppose inductively $a_n < 2$ for some $n \geq 1$. Consider,

$$a_{n+1} = \sqrt{a_n + 2} < \sqrt{2+2} = \sqrt{4} = 2.$$

Thus $a_n < 2$ $\forall n \in \mathbb{N}$ by PMI on n.

//

(b.) We prove $a_{n+1} \geq a_n$ by induction on n. Observe, $a_2 \geq a_1$ so $\sqrt{\sqrt{2}+2} \geq \sqrt{2}$. Suppose inductively that $a_{n+1} \geq a_n$ for some $n \in \mathbb{N}$. Notice $a_{n+1} + 2 \geq a_n + 2$ thus $\sqrt{a_{n+1} + 2} \geq \sqrt{a_n + 2}$ and hence $a_{n+2} \geq a_{n+1}$. Thus $\{a_n\}$ is increasing sequence.

(c.) $\{a_n\}$ is an increasing sequence bounded above by 2 thus $a_n \to a$ by Bounded Monotonic Sequence Thm.

Notice $\lim\limits_{n\to\infty} (a_{n+1}) = \lim\limits_{n\to\infty} \sqrt{a_n + 2} \Rightarrow a = \sqrt{a+2}$

thus $a^2 = a + 2 \Rightarrow a^2 - a - 2 = (a-2)(a+1) = 0 \Rightarrow a = 2$ or $a = -1$.

But, $a_n \geq 0$ by construction hence $\lim\limits_{n\to\infty} (a_n) \geq 0$

and we find $\boxed{a = 2}$.

P50 Ex 2.3.3 part a

Find and prove the limit of $\sqrt{2}$, $\sqrt{2\sqrt{2}}$, $\sqrt{2\sqrt{2\sqrt{2}}}$, ...

Let $a_1 = \sqrt{2}$ and recursively define $a_{n+1} = \sqrt{2a_n}$

Observe $a_1 = \sqrt{2} < 2$ thus $a_n < 2$ for $n = 1$.

Suppose inductively $a_n < 2$ for some $n \in \mathbb{N}$. Notice

$2a_n < 4$ implies $\sqrt{2a_n} < \sqrt{4} \Rightarrow a_{n+1} < 2$ $\therefore a_n < 2 \ \forall n \in \mathbb{N}$.

Furthermore, $a_2 > a_1$ since $\sqrt{2\sqrt{2}} > \sqrt{2}$. Suppose inductively

$a_{n+1} \geq a_n$ then $2a_{n+1} \geq 2a_n \Rightarrow \sqrt{2a_{n+1}} \geq \sqrt{2a_n}$

thus $a_{n+2} \geq a_{n+1}$ and we find (a_n) is increasing.

Therefore, by Bounded Monotonic Seq. Thm $\lim_{n\to\infty}(a_n) = a$.

Consider, $\lim_{n\to\infty}(a_{n+1}) = \lim_{n\to\infty}\sqrt{2a_n} \Rightarrow a = \sqrt{2a}$

and noting $a \neq 0$ is clear from $a_n \geq \sqrt{2}$ we

find $a^2 = 2a \Rightarrow a(a-2) = 0 \Rightarrow \underline{a = 2}$.

Thus $\sqrt{2}$, $\sqrt{2\sqrt{2}}$, ... converges to 2.

$\boxed{PS1}$ Ex 2.3.4 / $a_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \quad \forall n \in \mathbb{N}$.

Clearly $a_{n+1} = 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{1}{(n+1)!} > a_n$ thus $\{a_n\}$ inc.

Note $a_1 = 1 < 3$. Suppose inductively $a_n < 3$

and consider $a_{n+1} = 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{1}{(n+1)!} = a_n + \frac{1}{(n+1)!}$

thus $a_{n+1} = a_n + \frac{1}{(n+1)!} < 3 + \frac{1}{(n+1)!}$ then

$a_{n+1} - 3 < \frac{1}{(n+1)!}$ ... Sorry, I'm not seeing it currently.

Alternate approach

Cauchy Sequence can be characterized as one for which $\lim_{n\to\infty} (a_{n+p} - a_n) = 0$ following Remark 2.4.6 on p. 48.

$$a_{n+p} - a_n = \left(1 + \cdots + \frac{1}{n!} + \frac{1}{(n+1)!} + \cdots + \frac{1}{(n+p)!}\right) - \left(1 + \cdots + \frac{1}{n!}\right)$$

Therefore,

$$a_{n+p} - a_n = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots + \frac{1}{(n+p)!}$$

and it follows $\lim_{n\to\infty} (a_{n+p} - a_n) = 0$. Thus $\{a_n\}$ is Cauchy and by § 2.4 we find $\{a_n\}$ is convergent.

BTW, (from CALC II I know $\lim_{n\to\infty} \left(1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right) = e$)

P52 Ex. 2.3.6 part a / $\lim_{n\to\infty}\left(\frac{2n^2+n+1}{n-2}\right) = \infty$ prove via Defⁿ 2.3.2

Notice $n \geq 3$ by assumption given the expression $\frac{2n^2+n+1}{n-2}$.

Let $M > 0$ and suppose $n > \max\{3, M/2\}$. Consider, $2n > M$,

$$\frac{2n^2+n+1}{n-2} > \frac{2n^2}{n-2} > \frac{2n^2}{n} = 2n > M$$

thus $\lim_{n\to\infty}\left(\frac{2n^2+n+1}{n-2}\right) = \infty$.

P53 Ex 2.4.1 / Determine if Cauchy

(a.) $a_n = (-1)^n$ has $a_{2k} = 1$ and $a_{2k+1} = -1$ thus $\{a_n\}$ not convergent hence not Cauchy.

(b.) $a_n = \frac{(-1)^n}{n}$ $\quad \frac{-1}{n} \leq \frac{(-1)^n}{n} \leq \frac{1}{n}$ and $\frac{\pm 1}{n} \to 0$ hence by Squeeze $\lim_{n\to\infty}(a_n) = 0 \Rightarrow \{a_n\}$ convergent $\Rightarrow \{a_n\}$ Cauchy.

(c.) $a_n = \frac{n}{n+1}$ $\quad$ Note $\lim_{n\to\infty}\left(\frac{n}{n+1}\right) = \lim_{n\to\infty}\left(\frac{1}{1+1/n}\right) = 1$

Thus $\{a_n\}$ is convergent $\Rightarrow \{a_n\}$ is Cauchy.

(d.) $a_n = \frac{\cos(n)}{n}$ $\quad \frac{-1}{n} \leq \frac{\cos(n)}{n} \leq \frac{1}{n} \Rightarrow \lim_{n\to\infty}\left(\frac{\cos(n)}{n}\right) = 0$

$\Rightarrow \{a_n\}$ convergent

$\Rightarrow \{a_n\}$ is Cauchy.

It'd probably be more interesting to study $|a_m - a_n|$ for $m > n$,

(a.) $|a_m - a_n| = |(-1)^m - (-1)^n| = 1$ or 0

(b.) $|a_m - a_n| = \left|\frac{(-1)^m}{m} - \frac{(-1)^n}{n}\right|$

(c.) $|a_m - a_n| = \left|\frac{m}{m+1} - \frac{n}{n+1}\right| = \left|\frac{1}{1+1/m} - \frac{1}{1+1/n}\right|$

(d.) $|a_m - a_n| = \left|\frac{\cos(m)}{m} - \frac{\cos(n)}{n}\right|$

P54 Ex 2.4.2 / Prove $\frac{n\cos(3n^2+2n+1)}{n+1}$ has a convergent subseq.

By Bolzano Weierstrass Thm it suffices to show $a_n = \frac{n\cos(3n^2+2n+1)}{n+1}$ is bounded. Notice

$$-1 \le \cos(3n^2+2n+1) \le 1$$

and

$$\frac{-n}{n+1} \le \frac{n\cos(3n^2+2n+1)}{n+1} \le \frac{n}{n+1}$$

Notice $\frac{-n}{n+1} \longrightarrow -1$ and $\frac{n}{n+1} \longrightarrow 1$ hence $\{\frac{-n}{n+1}\}$ and $\{\frac{n}{n+1}\}$ are bounded. Suppose $\alpha < \frac{-n}{n+1}$ and $\frac{n}{n+1} < \beta$ $\forall n \in \mathbb{N}$. Let $M = \max\{|\alpha|, |\beta|\}$ and note $-M < \alpha$ and $\beta < M$

thus $-M < \alpha < \frac{-n}{n+1} \le \frac{n\cos(3n^2+2n+1)}{n+1} \le \frac{n}{n+1} < \beta < M$

$\Rightarrow |a_n| < M \;\; \forall n \in \mathbb{N} \;\therefore\; \{a_n\}$ bounded.

P55 Ex 2.4.4 / $a_n = \frac{1+2^n}{2^n}$ for $n \in \mathbb{N}$. Show a_n contractive

$$|a_{n+2} - a_{n+1}| = \left|\frac{1+2^{n+2}}{2^{n+2}} - \frac{1+2^{n+1}}{2^{n+1}}\right| = \frac{1}{2}\left|\frac{1+2^{n+2}}{2^{n+1}} - \frac{1+2^{n+1}}{2^n}\right|$$

$$= \frac{1}{2}\left|\frac{1+2^{n+2} - 2 - 2^{n+2}}{2^{n+1}}\right|$$

$$= \frac{1}{2}\left| \cdots \right|$$

I tried, now I'll try again

$\boxed{PS5}$ Ex 2.4.4. $a_n = \frac{1+2^n}{2^n}$ for $n \in \mathbb{N}$. Show a_n contractive

$$a_1 = \frac{1+2}{2} = \frac{3}{2} \quad \& \quad a_2 = \frac{1+2^2}{2^2} = \frac{5}{4} \quad \& \quad a_3 = \frac{1+8}{8} = \frac{9}{8} \text{ etc...}$$

$$|a_2 - a_1| = \left|\frac{5}{4} - \frac{3}{2}\right| = \left|\frac{5}{4} - \frac{6}{4}\right| = \frac{1}{4}$$

$$|a_3 - a_2| = \left|\frac{9}{8} - \frac{5}{4}\right| = \left|\frac{9-10}{8}\right| = \frac{1}{8} = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{2}|a_2 - a_1|$$

Let's try the algebra again,

$$|a_{n+2} - a_{n+1}| = \left|\frac{1+2^{n+2}}{2^{n+2}} - \frac{1+2^{n+1}}{2^{n+1}}\right|$$

$$= \left|\frac{1+2^{n+2} - 2 - 2^{n+2}}{2^{n+2}}\right|$$

$$= \left|\frac{-1}{2^{n+2}}\right|$$

$$= \frac{1}{2}\left|\frac{1}{2^{n+1}}\right|$$

However,

$$|a_{n+1} - a_n| = \left|\frac{1+2^{n+1}}{2^{n+1}} - \frac{1+2^n}{2^n}\right|$$

$$= \left|\frac{1+2^{n+1} - 2 - 2^{n+1}}{2^{n+1}}\right|$$

$$= \left|\frac{1}{2^{n+1}}\right|$$

Thus, $|a_{n+2} - a_{n+1}| \leq \frac{1}{2}|a_{n+1} - a_n|$.

$\boxed{P56}$ Ex. 2.4.5 Let $r \in \mathbb{R}$ such that $|r| < 1$. Let $a_n = r^n$ for $n \in \mathbb{N}$.

Consider $a_2 - a_1 = r^2 - r = r(r - r^*)$

$a_3 - a_2 = r^3 - r^2 = r^2(r-1) = r(r(r-1)) = r(a_2 - a_1)$

We suspect the contractive constant is $|r|$. Indeed,

$$|a_{n+2} - a_{n+1}| = |r^{n+2} - r^{n+1}| = |r||r^{n+1} - r^n| = |r||a_{n+1} - a_n|.$$

$\boxed{P57}$ Ex. 2.5.1 a & d find limsup & liminf

(a.) $a_n = (-1)^n$. Notice $\{a_k \mid k > n\} = \{1, -1\}$ for all n. Thus,

$$\limsup_{n\to\infty}(a_n) = \lim_{n\to\infty}\left(\sup\{a_k \mid k > n\}\right) = \lim_{n\to\infty}\left(\sup\{1,-1\}\right) = \lim_{n\to\infty}(1) = \boxed{1}$$

$$\liminf_{n\to\infty}(a_n) = \lim_{n\to\infty}\left(\inf\{a_k \mid k > n\}\right) = \lim_{n\to\infty}\left(\inf\{1,-1\}\right) = \lim_{n\to\infty}(-1) = \boxed{-1}$$

(d.) $a_n = n\sin\left(\frac{n\pi}{2}\right)$

$$a_{4k+1} = (4k+1)\sin\left(\frac{(4k+1)\pi}{2}\right) = (4k+1)\sin\left(2k\pi + \frac{\pi}{2}\right) = 4k+1$$

$$a_{4k-1} = (4k-1)\sin\left(\frac{(4k-1)\pi}{2}\right) = (4k-1)\sin\left(2k\pi - \frac{\pi}{2}\right) = -(4k-1)$$

thus $\{a_n\}$ is unbounded both above & below

hence $\limsup_{n\to\infty}(a_n) = \infty$ & $\liminf_{n\to\infty}(a_n) = -\infty$.

$\boxed{P58}$ Ex 2.5.2 part a (OMIT)

$\boxed{P59}$ Ex 2.5.4 OMIT.

$\boxed{P60}$ 2.2.5.

(a.) true, proof ↷

(b.) $\{n\}$ and $\{-n\}$ both diverge but $d_n = n - n = 0$ converges.
Thus $\{a_n\}, \{b_n\}$ divergent $\not\Rightarrow$ $\{a_n + b_n\}$ divergent.

(c.) $\lim_{n\to\infty}(a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$ thus $\{a_n\}, \{b_n\}$ conv. $\Rightarrow$ $\{a_n b_n\}$ conv.

↷

(d.) $a_n = (-1)^n$ and $b_n = (-1)^n$ are divergent sequences
yet $a_n b_n = (-1)^n(-1)^n = (-1)^{2n} = ((-1)^2)^n = 1^n = 1.$
Hence the product of divergent seq. need not be divergent.

(e.) Suppose $\{a_n\}$ and $\{a_n + b_n\}$ are convergent then
$b_n = a_n + b_n - a_n$ and by linearity of the limit,

$$\lim_{n\to\infty} (b_n) = \lim_{n\to\infty} (a_n + b_n) - \lim_{n\to\infty} (a_n) \in \mathbb{R}.$$

Thus $\{b_n\}$ is convergent.

(f.) $\{a_n\}$ & $\{a_n + b_n\}$ divergent $\not\Rightarrow$ $\{b_n\}$ divergent.
Consider the following counter-example,
$a_n = n$ and $b_n = 1$ then $\underbrace{a_n + b_n = n+1 \text{ \& } a_n = n}_{\text{divergent}}$

yet $b_n = 1$ is convergent.

Proof of (b.) Of course, this is a limit law, we've proved in lecture.
If $a_n \to a$ and $b_n \to b$ then for $\varepsilon > 0$ $\exists N_a, N_b \in \mathbb{N}$
for which $n \geq N_a \Rightarrow |a_n - a| < \varepsilon/2$ & $n \geq N_b \Rightarrow |b_n - b| < \varepsilon/2$.
Let $N = \max\{N_a, N_b\}$ then $n \geq N$ provides,

$$\begin{aligned} |a_n + b_n - (a+b)| &= |a_n - a + b_n - b| \\ &\leq |a_n - a| + |b_n - b| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Thus $a_n + b_n \to a+b$ and thus $\{a_n + b_n\}$ converges.