The text for this portion of the course are the pdf's provided in Course Content (they will appear later in the term). This homework covers the material discussed in Lectures 23, 24, 25, 26, 27 and 28. It is due 11-20-20.

Problem 101: Apply Newton's method to the function $f(x) = x^2 - a$ where $a > 0$. Show $x_{n+1} = \frac{1}{2}(x_n + a/x_n)$. If $x_o > \sqrt{a}$, show that the sequence $\{x_n\}_{n=0}^{\infty}$ converges to $\sqrt{a}$ by proving that $\phi(x) = \frac{1}{2}(x + a/x)$ is a contraction mapping of $[\sqrt{a}, x_o]$.

Problem 102: Use the problem above to calculate $\sqrt{2}$ accurately to three decimal places[1].

Problem 103: Prove that the equation $2 - x - \sin x = 0$ has a unique real root, and that it lies in the interval $[\pi/6, \pi/2]$. Show that $\phi(x) = 2 - \sin(x)$ is a contraction mapping of this interval.

Problem 104: Continuing the previous problem, apply an appropriate theorem to find the root, accurate to three decimal places[2].

Problem 105: Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ have formula $T(x, y) = (x + y, x - y, 3y)$. Find the standard matrix $[T]$ of this linear transformation.

Problem 106: Let $S : \mathbb{R}^3 \to \mathbb{R}$ have formula $S(x, y, z) = x + 2y + 3z$. Find the standard matrix $[S]$ of this linear transformation.

Problem 107: Continuing the above, form $S \circ T : \mathbb{R}^2 \to \mathbb{R}$ where $(S \circ T)(x, y) = S(T(x, y))$ for all $(x, y) \in \mathbb{R}^2$. Explicitly calculate the formula for $S \circ T$ and then find its standard matrix $[S \circ T]$. Check that $[S \circ T] = [S][T]$ where we indicate matrix multiplication in the right-hand-side of the equation.

Problem 108: Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x, y) = (x^2 + y^2, xy)$.
Find the linearization of F at $(1, 3)$.

Problem 109: Suppose $F : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x_o \in \mathbb{R}^n$. Prove F is continuous at x_o.

Problem 110: Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x, y) = (x^2 - y^2, 2xy)$. Calculate the Jacobian matrix of F and check where the Jacobian matrix is invertible. *Since this function is continuously differentiable on $\mathbb{R}^2$ it follows that each point where J_F is invertible has a neighborhood on which F is locally invertible.*

Problem 111: Find the inverse map to F given in the previous problem. The way I would attempt the calculation is to set $x^2 - y^2 = a$ and $2xy = b$ then work on solving these equations for $x = G_1(a, b)$ and $y = G_2(a, b)$. Then $G = (G_1, G_2)$ should serve as the inverse map to F.

Problem 112: Still stuck on the previous problem ? Maybe try changing to polar coordinates, perhaps it is easier to solve in terms of r, θ then convert back to x, y.

Problem 113: Consider the map $F(x, y, z) = (x/y, y/z, z/x)$. Calculate the Jacobian J_F and find all singular points of J_F.

[1] this problem and the preceding one are based on Edward's Advanced Calculus page 171, problem 1.6.
[2] this problem and the preceding one are based on Edward's Advanced Calculus page 171, problem 1.7

Problem 114: Use algebra to find the inverse map to F in the previous problem. How does the algebra compare to the singularities in the Jacobian in the previous problem ?

Problem 115: Find the multivariate power series centered at $(0,0)$ for $f(x,y) = x\sin(y) + cos(x)$.

Problem 116: Find the multivariate power series centered at $(1,-1)$ for $f(x,y) = e^{xy}$

Problem 117: Consider the quadratic form $Q(x,y) = 10x^2 - 8xy + 10y^2$. Find the eigenvalues of the quadratic form. Write the equation on $Q(x,y) = 1$ in eigencoordinates and identify the curve.

Problem 118: Consider the quadratic form $Q(x,y,z) = 31x^2 + 15y^2 + 15z^2 - 22xy - 22xz + 10yz$. Find the eigenvalues of the quadratic form. Write the equation on $Q(x,y,z) = 1$ in eigencoordinates and identify the surface. (please use technology to find the eigenvalues, I don't expect you to calculate this one brute-force)

Problem 119: Let $f(x,y) = 3 + 10(x-1)^2 - 4(x-1)(y-2) + 6(y-2)^2 + \cdots$. Determine if $f(1,2) = 3$ is a minimum, maximum or a saddle point in view of the given multivariate Taylor expansion.

Problem 120: Consider $f(x,y,z) = \dfrac{1+y^2}{1-2xz}$. Show $(0,0,0)$ is a critical point of f and calculate the Hessian of f at $(0,0,0)$. Find the spectrum of the Hessian and use it to classify the critical point as maximum, minimum or saddle-type.