Show work for full credit. 300pts to earn here. Please write using a pen or pencil which scans well.

(1.) (20pts) Let A, B, C, D be sets. Suppose $A \subseteq B$ and $C \subseteq D$. Prove $A \times C \subseteq B \times D$.

(2.) (25pts) Let $f(x) = 1 + e^x$. Find:

(a.) $f([0, \ln(2)])$

(b.) $f^{-1}([0, 3])$

(3.) (25pts) Let $f(x) = 3(x-1)^2$ define $f : [1, \infty) \to [0, \infty)$. Prove f is a bijection. That is, show f is both injective and surjective.

(4.) (30pts) Suppose $a \neq 1$. Prove $1 + a + a^2 + \cdots + a^n = \dfrac{1 - a^{n+1}}{1 - a}$ for all $n \in \mathbb{N}$.

(5.) (20pt) Let $S = \{1 + \frac{(-1)^n}{n} \mid n \in \mathbb{N}\}$. Find $\inf(S)$ and $\sup(S)$. No need for proof of your claims here, but beware partial credit cannot be given if you're incorrect and show no work.

(6.) (20pts) Let $S = \{x \in \mathbb{R} \mid 0 < x < 1\} \cup \{1, 2, 3\}$. Find each isolated point of S. Also, explain which points in $\mathbb{R} - S$ are limit points of S.

(7.) (20pts) Suppose A, B are compact subsets of $\mathbb{R}$. Prove $A \cup B$ is compact.

(8.) (30pts) Prove $\lim_{x \to 2}(x^3 - 1) = 7$ by an explicit ϵ, δ-style argument.

(9.) (30pts) Suppose $a \neq 0$ and $b \in \mathbb{R}$. Define $f(x) = ax + b$ for each $x \in \mathbb{R}$. Prove f is continuous by an explicit ϵ, δ-argument based on the defn. of continuity.

(10.) (20pts) Prove $a_n = \frac{n \cos(4n^2+2)}{n+3}$ has a convergent subsequence.

(11.) (20pts) Let $f : [a, b] \to [3a, 3b]$ be a continuous function. Prove $f(x) = 3x$ has a **unique** solution on $[a, b]$.

(12.) (20pts) Prove $\frac{d}{dx} \sin x = \cos x$ given that $\lim_{h \to 0} \frac{\sin(h)}{h} = 1$ and $\lim_{h \to 0} \frac{1 - \cos(h)}{h} = 0$.

(13.) (20pts) Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $F(x, y, z) = (x^3, x + 2y + 3z, x + y + z)$.
Find the linearization of F at (x_o, y_o, z_o). At which points does the inverse mapping theorem indicate the existence of a local inverse for F ?