MH 408: Topics in Analysis:	Test 3

Show work for full credit. At least 100pts to earn here.

(1.) (10pts) Prove $\frac{d}{dx}\cos x = -\sin x$ given that $\lim_{h\to 0}\frac{\sin(h)}{h} = 1$ and $\lim_{h\to 0}\frac{1-\cos(h)}{h} = 0$.

(2.) (10pts) Exercise 4.2.1 from the text.

(3.) (10pts) Use the Mean Value Theorem to prove $e^x > 1 + x$ for $x > 0$. Assume we know $\frac{d}{dx}e^x = e^x$.

(4.) (10pts) Exercise 4.3.5 from the text.

(5.) (10pts) Exercise 4.5.3 from the text.

(6.) (10pts) Prove that the equation $2 - x - \sin x = 0$ has a unique real root, and that it lies in the interval $[\pi/6, \pi/2]$. Show that $\phi(x) = 2 - \sin(x)$ is a contraction mapping of this interval.

(7.) (10pts) Let $T : \mathbb{R}^4 \to \mathbb{R}^2$ be defined by $T(a, b, c, d) = (a + 2b + 3c, 4b + 5c + 6d)$ for each $(a, b, c, d) \in \mathbb{R}^4$. Find the standard matrix for T.

(8.) (15pts) Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $F(x, y, z) = (x^2 + y^2 + z^2, xyz, x + y + z)$. Find the linearization of F at (x_o, y_o, z_o). At which points does the inverse mapping theorem indicate the existence of a local inverse for F ?

(9.) (15pts) Consider the quadratic form $Q(x, y) = 4x^2 + 2xy + 4y^2$. Find the eigenvalues of the quadratic form. Write the equation on $Q(x, y) = 1$ in eigencoordinates and identify the curve.

(10.) (10pts) Let $f(x, y, z) = 4 + 4(x - 1)^2 + 2(x - 1)(y - 2) + 4(y - 2)^2 + 4(z - 3)^2 + \cdots$. Determine if $f(1, 2, 3) = 4$ is a minimum, maximum or a saddle point in view of the given multivariate Taylor expansion.

TEST 3 SOLUTION:

$\boxed{P1}$ $$\frac{d}{dx}\left[\cos(x)\right] = \lim_{h\to 0}\left[\frac{\cos(x+h)-\cos(x)}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x)}{h}\right]$$

$$= \lim_{h\to 0}\left[\cos(x)\left(\frac{\cos(h)-1}{h}\right) - \sin(x)\frac{\sin(h)}{h}\right]$$

$$= \cos(x)\underbrace{\lim_{h\to 0}\left(\frac{\cos(h)-1}{h}\right)}_{0} - \sin(x)\underbrace{\lim_{h\to 0}\left(\frac{\sin(h)}{h}\right)}_{1}$$

$$= -\sin(x). //$$

$\boxed{P2}$ Suppose f and g are differentiable at x_0 and $f(x_0) = g(x_0)$ and $f(x) \leq g(x)\ \forall x \in \mathbb{R}$. Prove $f'(x_0) = g'(x_0)$

Let $h(x) = g(x) - f(x)$ and observe $h(x) \geq 0\ \forall x \in \mathbb{R}$. However, $h(x_0) = g(x_0) - f(x_0) = 0$ thus x_0 is a critical value and by Fermat's Thm we find $h'(x_0) = 0$. But, $h'(x) = g'(x) - f'(x) \Rightarrow g'(x_0) - f'(x_0) = 0 \therefore f'(x_0) = g'(x_0).//$

$\boxed{P3}$ Let $f(x) = e^x$ then $f'(x) = e^x > 0$ hence f increases on $\mathbb{R}$. Suppose $0 < x$ then the Mean Value Thm provides $c \in (0, x)$ for which

$$\frac{f(x)-f(0)}{x-0} = f'(c) \Rightarrow \frac{e^x - e^0}{x} = \underbrace{e^c > e^0}_{\text{since } f \text{ increases}} = 1$$

Thus $$\frac{e^x - 1}{x} > 1 \Rightarrow e^x - 1 > x \Rightarrow e^x > 1 + x \text{ for } x > 0.//$$

P4 Let f be twice differentiable on open interval I. Suppose $\exists\, a, b, c \in I$ with $a < b < c$ such that $f(a) < f(b)$ and $f(b) > f(c)$. Prove $\exists\, d \in (a, c)$ s.t. $f''(d) < 0$.

Apply MVT to f on $[a, b]$ to find $m_1 \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(m_1).$$ Observe $f(a) < f(b) \Rightarrow f'(m_1) > 0$.

Likewise, apply MVT to f on $[b, c]$ to obtain $m_2 \in (b, c)$ for which $\dfrac{f(c) - f(b)}{c - b} = f'(m_2)$. Note $f'(m_2) < 0$ since $f(b) > f(c)$. Since f is twice diff. we have f' as a diff. function on $[m_1, m_2]$ hence apply MVT to f' to obtain $d \in (m_1, m_2) \subseteq (a, c)$ s.t.

$$\frac{f'(m_2) - f'(m_1)}{m_2 - m_1} = f''(d)$$

but $f'(m_2) < 0$ and $f'(m_1) > 0 \Rightarrow f''(d) < 0$. //

P5 Apply Th$^{\underline{m}}$ 4.5.3 to decide local max or min at $\bar{x}$

(a.) $\bar{x} = 0$, $f(x) = x^3 \sin(x)$

$$f'(x) = 3x^2 \sin x + x^3 \cos x$$

$$f''(x) = 6x \sin x + 3x^2 \cos x + 3x^2(\cos x) - x^3 \sin x$$

$$f''(x) = (6x - x^3)\sin(x) + (3x^2 + 3x^2)\cos x$$

$$f'''(x) = (6 - 3x^2)\sin x + (6x - x^3)\cos(x) + 12x\cos x - 6x^2 \sin x$$

$$f'''(x) = (6 - 9x^2)\sin x + (18x - x^3)\cos(x)$$

$$f^{(4)}(x) = -18x \sin x + (6 - 9x^2)\cos(x) + (18 - 3x^2)\cos(x) - (18x - x^3)\sin x$$

Observe $f'(0) = f''(0) = f'''(0) = 0$ but $f^{(4)}(0) = 6 + 18 = 24 > 0$

Thus $f(0) = 0$ serves as a local minimum by Th$^{\underline{m}}$ 4.5.3.

PS continued (Ex 4.5.3 continued)

(6.) $\bar{x} = 1$, $f(x) = (1-x)\ln(x)$

$$f'(x) = -\ln(x) + \frac{1-x}{x} = \frac{1}{x} - \ln(x) - 1$$

$$f''(x) = \frac{-1}{x^2} - \frac{1}{x}$$

$$f'''(x) = \frac{2}{x^3} + \frac{1}{x^2}$$

Observe $f'(1) = 0$ and $f''(1) = -2 < 0$ thus $f(1) = 0$ is local max

$\boxed{P6}$ Consider $\varphi(x) = 2 - \sin(x)$ and $\varphi : [\pi/6, \pi/2] \longrightarrow \mathbb{R}$

Observe $\varphi'(x) = -\cos(x) < 0$ for $\frac{\pi}{6} \leq x < \frac{\pi}{2}$ thus φ is decreasing on $[\pi/6, \pi/2)$. Observe for $c_2 \in [\pi/6, \pi/2)$ we have

$$\varphi(\pi/6) > \varphi(c_2)$$

Thus $\varphi(\frac{\pi}{6}) = 2 - \sin(\frac{\pi}{6}) = 2 - \frac{1}{2} = \frac{3}{2}$ is upper bound on images of φ. On the other extreme, $\varphi(\frac{\pi}{2}) = 2 - 1 = 1$ is the minimum value attained by φ thus $1 \leq \varphi(x) \leq \frac{3}{2}$ for $x \in [\pi/6, \pi/2]$

This implies $\varphi([\pi/6, \pi/2]) \subseteq [1, \frac{3}{2}] \subseteq [\frac{\pi}{6}, \frac{\pi}{2}]$. Let $x, y \in [\pi/6, \pi/2)$ with $x < y$ then by MVT, $\exists\, c \in (x, y)$ with

$$\frac{\varphi(y) - \varphi(x)}{y - x} = \varphi'(c) = -\cos(c)$$

Thus $|\varphi(y) - \varphi(x)| = |\cos(c)||y - x| \leq |y - x|$ (not helpful!)

Notice $\frac{\pi}{6} < c < \frac{\pi}{2}$ hence $\cos(\frac{\pi}{6}) > \cos(c) > \cos(\frac{\pi}{2}) \Rightarrow \frac{\sqrt{3}}{2} > \cos(c)$

thus $|\varphi(y) - \varphi(x)| \leq \frac{\sqrt{3}}{2}|y - x|$.

P6 continued

We've shown $\varphi(x) = 2 - \sin(x)$ is contraction map on $[\pi/6, \pi/2]$ with contraction constant $k = \frac{\sqrt{3}}{2} < 1$.

Thus by contraction mapping Th^{m} $\exists!\, x_0 \in [\pi/6, \pi/2]$ for which $\varphi(x_0) = x_0$ which means

$x_0 = 2 - \sin(x_0) \Rightarrow 2 - x - \sin x = 0$ has unique sol^{n} x_0.//

$\boxed{P7}$ $T(a,b,c,d) = (a + 2b + 3c,\ 4b + 5c + 6d)$

$$\left.\begin{aligned} T(1,0,0,0) &= (1,0) \\ T(0,1,0,0) &= (2,4) \\ T(0,0,1,0) &= (3,5) \\ T(0,0,0,1) &= (0,6) \end{aligned}\right\} \quad \boxed{[T] = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 4 & 5 & 6 \end{bmatrix}}$$

$\boxed{P8}$ $F(x,y,z) = (x^2 + y^2 + z^2,\ xyz,\ x + y + z)$

$$J_F = \left[\frac{\partial F}{\partial x} \middle| \frac{\partial F}{\partial y} \middle| \frac{\partial F}{\partial z} \right] = \begin{bmatrix} 2x & 2y & 2z \\ yz & xz & xy \\ 1 & 1 & 1 \end{bmatrix}$$

notice $x = y = z$ makes these column dependent.

Thus

$$\boxed{\begin{aligned} L_F(x,y,z) = (x_0^2 + y_0^2 + z_0^2,\ x_0 y_0 z_0,\ x_0 + y_0 + z_0) &+ \begin{bmatrix} 2x_0 & 2y_0 & 2z_0 \\ y_0 z_0 & x_0 z_0 & x_0 y_0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix}. \end{aligned}}$$

$$\det(J_F) = 2x \det\begin{bmatrix} xz & xy \\ 1 & 1 \end{bmatrix} - 2y \det\begin{bmatrix} yz & xy \\ 1 & 1 \end{bmatrix} + 2z \det\begin{bmatrix} yz & xz \\ 1 & 1 \end{bmatrix}$$

$$= 2x(xz - xy) - 2y(yz - xy) + 2z(yz - xz)$$

$$= 2x^2(z - y) + 2y^2(x - z) + 2z^2(y - x) \neq 0$$

Points with $\det(J_F) \neq 0$ have local inverse's existence given by IFT.

$\boxed{P9}$ $Q(x,y) = 4x^2 + 2xy + 4y^2 \to [Q] = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$

$\det \begin{bmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{bmatrix} = (\lambda-4)^2 - 1 = (\lambda-5)(\lambda-3) = 0$

thus $\lambda_1 = 3$ & $\lambda_2 = 5 \Rightarrow Q(x,y) = 3\bar{x}^2 + 5\bar{y}^2$

for eigen coordinates $\bar{x}, \bar{y}$ thus $Q(x,y) = 1$

is simply $3\bar{x}^2 + 5\bar{y}^2 = 1$ an $\boxed{\text{Ellipse}}$.

$\boxed{P10}$ $f(x,y,z) = 4 + 4(x-1)^2 + 2(x-1)(y-2) + 4(y-2)^2 + 4(z-3)^2 + \cdots$

Hessian matrix,

$$\mathcal{H} = [Q] = [\partial_i \partial_j f] = \left[\begin{array}{cc|c} 4 & 1 & 0 \\ 1 & 4 & 0 \\ \hline 0 & 0 & 4 \end{array}\right]$$

$\leftarrow$ I can see answer from $\boxed{P9}$ and theory of eigenvalues...

It follows $\lambda_1 = 3$, $\lambda_2 = 4$, $\lambda_3 = 5$. Thus

$$f(x,y,z) = 4 + 3\bar{x}^2 + 4\bar{y}^2 + 5\bar{z}^2 + \cdots$$

in eigen coordinates for $x-1, y-2, z-3$ w.r.t. $[Q]$

thus $f(x,y,z) = 4$ at $(1,2,3)$ is <u>local</u> <u>minimum</u>