

MATH 408: CALCULUS OF HIGHER DIMENSION : TEST 1 SOLUTION

(1.)  $\vec{A} = \langle 1, 2, 2 \rangle$  and  $\vec{B} = \langle 1, 0, -1 \rangle$

(a.)  $\vec{A} \cdot \vec{B} = 1 \cdot 1 + 2 \cdot 0 + 2 \cdot (-1) = 1 - 2 = \boxed{-1}$

(b.) since  $A = \sqrt{1+2^2+2^2} = \sqrt{9} = 3$  and  $B = \sqrt{1^2+0^2+(-1)^2} = \sqrt{2}$   
 and  $\vec{A} \cdot \vec{B} = AB \cos \theta \Rightarrow \theta = \cos^{-1} \left( \frac{-1}{3\sqrt{2}} \right) = \boxed{103.63^\circ}$

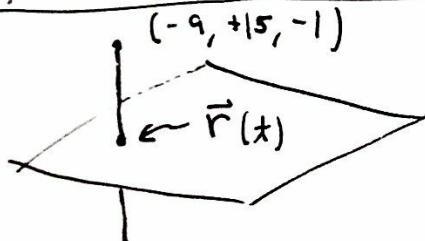
(c.)  $\vec{A} \cdot \hat{g} = \langle 1, 2, 2 \rangle \cdot \langle 0, 1, 0 \rangle = 2 = 3 \cos \beta \Rightarrow \beta = \cos^{-1} \left( \frac{2}{3} \right) = \boxed{48.19^\circ}$

(d.)  $\text{Proj}_{\vec{B}}(\vec{A}) = (\vec{A} \cdot \hat{B}) \hat{B} = \frac{1}{B^2} (\vec{A} \cdot \vec{B}) \vec{B} = \frac{1}{2} (-1) \langle 1, 0, -1 \rangle = \boxed{\langle -\frac{1}{2}, 0, \frac{1}{2} \rangle}$

(e.)  $\vec{A} \times \vec{B} = \det \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 2 & 2 \\ 0 & 0 & -1 \end{bmatrix} = \boxed{\langle -2, 3, -2 \rangle}$  ← notice as  $(\vec{A} \times \vec{B}) \cdot \vec{A} = 0$   
 a check,  $(\vec{A} \times \vec{B}) \cdot \vec{B} = 0$

(2.)  $\vec{A} \perp \vec{B}$  then  $\vec{A} \cdot \vec{B} = 0$ . Notice  $\vec{A} \cdot \vec{A} = \|\vec{A}\|^2$ ,  $\vec{B} \cdot \vec{B} = \|\vec{B}\|^2$ . Likewise,  
 $\|\vec{A} + \vec{B}\|^2 = (\vec{A} + \vec{B}) \cdot (\vec{A} + \vec{B}) = \vec{A} \cdot \vec{A} + \vec{A} \cdot \vec{B} + \vec{B} \cdot \vec{A} + \vec{B} \cdot \vec{B} = \|\vec{A}\|^2 + \|\vec{B}\|^2$ .

(4.) Find point on plane  $3x - 5y + 2z = 10$  closest to  $(-9, 15, -1)$



Let  $\vec{r}(t) = \langle \underbrace{-9+3t}_x, \underbrace{-5t+15}_y, \underbrace{-1+2t}_z \rangle$

$10 = 3x - 5y + 2z = 3(-9+3t) - 5(15-5t) + 2(-1+2t)$

$9t + 25t + 4t = 10 + 27 + 75 + 2$

$38t = 114$

$t = 3 \Rightarrow \boxed{\vec{r}(3) = (0, 0, 5)}$

← closest point.

(5.)  $\vec{r}(t) = \langle \sin t, e^t, \cos t \rangle$

$\frac{d\vec{r}}{dt} = \vec{v} = \langle \cos t, e^t, -\sin t \rangle \Rightarrow v = \sqrt{\cos^2 t + e^{2t} + \sin^2 t}$

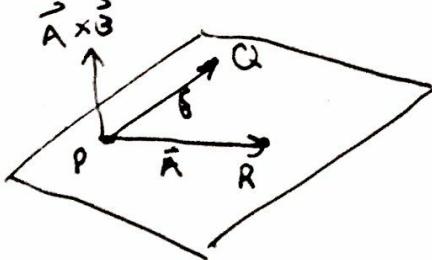
$\therefore v = \sqrt{1+e^{2t}} = \dot{s}$  (speed)

$a_T = \ddot{s} = \frac{d}{dt} \left( \sqrt{1+e^{2t}} \right)$

$= \frac{2e^{2t}}{2\sqrt{1+e^{2t}}}$

$\boxed{a_T = \frac{e^{2t}}{\sqrt{1+e^{2t}}}}$

$$(3.) P = (0, 1, 5), Q = (3, 3, 7), R = (-4, 0, 14)$$



$$\vec{A} = R - P = \langle -4, -1, 9 \rangle$$

$$\vec{B} = Q - P = \langle 3, 2, 2 \rangle$$

$$\vec{A} \times \vec{B} = \det \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ -4 & -1 & 9 \\ 3 & 2 & 2 \end{bmatrix} = \underbrace{\langle -20, 35, -5 \rangle}_{\text{normal to plane.}}$$

Using basepoint  $(0, 1, 5)$ ,

$$-20x + 35(y-1) - 5(z-5) = 0 \quad \text{and} \quad \text{area} = \frac{1}{2} \|\vec{A} \times \vec{B}\| = \frac{5\sqrt{66}}{2} \approx 20.3$$

$$-20x + 35y - 5z = 10$$

6.)  $C_t: x = 3\cos t, -y = 3\sin t, 0 \leq t \leq \pi$   
 $\ell: x = -3 + 6t, y = 0, 0 \leq t \leq 1$   
 check it  $\begin{cases} t=0 \text{ gives } (-3, 0) \\ t=1 \text{ gives } (3, 0) \end{cases}$

$$f(x, y) = xy.$$

$$0 \leq f(x, 0) = x(0) = 0.$$

$$\int_C f ds = \int_{\ell} f ds + \int_{C_+} f ds = \int_0^{\pi} f(3\cos t, 3\sin t) \underbrace{\sqrt{(-3\sin t)^2 + (3\cos t)^2}}_{ds} dt$$

$$ds = \sqrt{x^2 + y^2} dt = \sqrt{9} dt = 3dt.$$

Thus,

$$\int_C f ds = \int_0^{\pi} 27 \cos t \sin t dt = \frac{27}{2} \sin^2(t) \Big|_0^{\pi} = 0.$$

(7.) (a.) Equation I. is an ellipsoid. (D.)

(b.) Eq. III. is a hyperboloid of two sheets (B.)

(c.) Eq. IV. is a hyperboloid of one sheet (A.)

(d.) Eq. II. is a cone (C.)

T, N, B 3pts each  
 T 2pts  
 N 2pts

$$8.) \vec{r}(t) = \langle 1 + 3\sin t, 2 + 4t, 3 + 3\cos t \rangle$$

$$\vec{v} = \langle 3\cos t, 4, -3\sin t \rangle \Rightarrow \dot{s} = v = \sqrt{9 + 16} = \sqrt{25} = 5. \quad \text{curvature}$$

$$\therefore T = \frac{1}{5} \langle 3\cos t, 4, -3\sin t \rangle$$

$$T' = \frac{1}{5} \langle -3\sin t, 0, -3\cos t \rangle \Rightarrow \|T'\| = \frac{3}{5} \quad \text{and} \quad B = \frac{\|T'\|}{v} = \frac{3/5}{5} = \frac{3}{25}$$

$$\therefore N = \hat{T}' = \langle -\sin t, 0, -\cos t \rangle = N$$

$$B = T \times N = \det \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ 3\cos t & 4 & -3\sin t \\ -\sin t & 0 & -\cos t \end{bmatrix} = \langle -\frac{4}{5} \cos t, \frac{3}{5}, \frac{4}{5} \sin t \rangle = B$$

$$\frac{dB}{dt} = \langle \frac{4}{5} \sin t, 0, \frac{4}{5} \cos t \rangle \quad N \cdot \frac{dB}{dt} = -\frac{4}{5} (\sin^2 t + \cos^2 t) = -\frac{4}{5} \Rightarrow T = \frac{4}{25}$$

$\uparrow$   
 torsion.

$$(9.) \quad \begin{aligned} T' &= v \kappa N \\ N' &= -v \kappa T + v \tau \theta \\ B' &= -v \tau N \end{aligned}$$

Derive:  $\kappa = \frac{\|\vec{r}' \times \vec{r}''\|}{\|\vec{r}'\|^3}$

$$\tau = \frac{(\vec{r}' \times \vec{r}'') \cdot \vec{r}''}{\|\vec{r}' \times \vec{r}''\|^2}$$

$$T = \frac{1}{\|\vec{r}'\|} \quad \vec{r}' = \frac{1}{v} \vec{v}$$

$$\vec{r}'' = \vec{v}' = \frac{d}{dt}(vT) = \frac{dv}{dt}T + v \frac{dT}{dt} = \frac{dv}{dt}T + v^2 \kappa N$$

$$\vec{r}' \times \vec{r}'' = \vec{r}' \times [ \frac{dv}{dt}T + v^2 \kappa N ]$$

$$= vT \times [ \frac{dv}{dt}T + v^2 \kappa N ]$$

$$= v \dot{v} (T \times T) + v^3 \kappa (T \times N)$$

$$= v^3 \kappa B \quad \therefore \|\vec{r}' \times \vec{r}''\| = v^3 \kappa \|\vec{B}\| = v^3 \kappa$$

$$\therefore \boxed{\kappa = \frac{\|\vec{r}' \times \vec{r}''\|}{v^3}}$$

$$\vec{r}''' = \ddot{v}T + \dot{v}T' + \frac{d}{dt}(v^2 \kappa)N + v^2 \kappa N'$$

$$\vec{r}''' = \ddot{v}T + \dot{v}\kappa v N + \frac{d}{dt}(v\kappa)N + v^2 \kappa (-v\kappa T + v\tau \theta)$$

But,  $\vec{r}' \times \vec{r}'' = v^3 \kappa B$  hence only the  $B$ -coefficient matters,

$$(\vec{r}' \times \vec{r}'') \cdot \vec{r}''' = (v^3 \kappa)(v^2 \kappa v T) B \cdot B = v^6 \kappa^2 T$$

$$\text{But, } \kappa^2 = \frac{\|\vec{r}' \times \vec{r}''\|}{v^3} \quad \therefore (\vec{r}' \times \vec{r}'') \cdot \vec{r}''' = v^6 T \left( \frac{\|\vec{r}' \times \vec{r}''\|}{v^3} \right)^2$$

$$\Rightarrow \boxed{T = \frac{(\vec{r}' \times \vec{r}'') \cdot \vec{r}''}{\|\vec{r}' \times \vec{r}''\|^2}} \quad \text{yep.}$$

(10.)  $\vec{r}(t) = \vec{r}_0 + \cos t \vec{b} + \sin t \vec{c}$  where  $\vec{b} \cdot \vec{c} = 0$  and  $\|\vec{b}\| = \|\vec{c}\| = 1$

$$\begin{aligned} \vec{v} &= -\sin t \vec{b} + \cos t \vec{c} \\ \vec{a} &= -\cos t \vec{b} - \sin t \vec{c} \end{aligned}$$

$$\Rightarrow v = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \quad (\text{using problem 2})$$

$$\therefore \boxed{T = \vec{v}}.$$

$$T' = \vec{v}' = \vec{a} = -\cos t \vec{b} - \sin t \vec{c}$$

$$\|T'\| = \|- \cos t \vec{b} - \sin t \vec{c}\| = 1.$$

$$\kappa = \frac{\|T'\|}{v} = 1 \quad \therefore \boxed{\kappa = 1}$$

(CIRCULAR MOTION)

Notice,

$$\begin{aligned} \vec{r}' \times \vec{r}'' &= \vec{v} \times \vec{a} \\ &= (-\sin t \vec{b} + \cos t \vec{c}) \times (-\cos t \vec{b} - \sin t \vec{c}) \\ &= \sin^2 t \vec{b} \times \vec{c} - \cos^2 t \vec{c} \times \vec{b} \\ &= \vec{b} \times \vec{c} = v^3 \kappa B = B. \end{aligned}$$