

Please box your answer. Show work for full credit. A scientific, non-graphing, calculator is allowed. You are also allowed a 3×5 card with whatever helps.

- (1.) (10pts) Let $f(x, y) = \sin(x^2y)$ calculate f_{xx} and f_{yy} and f_{xy} .

$$f_x = \frac{\partial}{\partial x} f = \frac{\partial}{\partial x} [\sin(x^2y)] = 2xy \cos(x^2y)$$

$$f_y = \frac{\partial}{\partial y} [\sin(x^2y)] = x^2 \cos(x^2y)$$

$$f_{xy} = \frac{\partial}{\partial x} (x^2 \cos(x^2y)) = \boxed{2x \cos(x^2y) - 2x^3y \sin(x^2y)} = f_{xy}$$

$$f_{xx} = y \frac{\partial}{\partial x} (2x \cos(x^2y)) = y \left[2 \cos(x^2y) - 2x \sin(x^2y) \frac{\partial}{\partial x} (x^2y) \right]$$

$$\therefore \boxed{f_{xx} = 2y \cos(x^2y) - 4x^3y^2 \sin(x^2y)}$$

$$f_{yy} = \frac{\partial}{\partial y} [x^2 \cos(x^2y)] = x^2 (-\sin(x^2y) \frac{\partial}{\partial y} (x^2y))$$

$$\therefore \boxed{f_{yy} = -x^4 \sin(x^2y)}$$

- (2.) (10pts) Calculate $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{x^2 + y^2}$ (if it doesn't exist then explain why)

Consider the path $(x, 0) \rightarrow (0, 0)$

$$\lim_{(x,0) \rightarrow (0,0)} \left(\frac{x^2}{x^2 + y^2} \right) = \lim_{x \rightarrow 0} \left(\frac{x^2}{x^2 + 0^2} \right) = \lim_{x \rightarrow 0} (1) = \boxed{1}$$

Likewise, $(0, y) \rightarrow (0, 0)$ gives

$$\lim_{(0,y) \rightarrow (0,0)} \left(\frac{x^2}{x^2 + y^2} \right) = \lim_{y \rightarrow 0} \left(\frac{0}{0 + y^2} \right) = \boxed{0}$$

Thus $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{x^2 + y^2}$ d.n.e. as we shown there exist two path limits which disagree ($0 \neq 1$).

(3.) (20pts) Let $f(x, y) = x^2 + 4xy + 6y^2$. For (a.) and/or (b.) leave answer (if any) as unit-vector with explicit decimal components to two decimal places.

- (a.) in what direction(s) does f change at rate 9 at $(2, 0)$?
- (b.) in what direction(s) is f constant at $(2, 0)$?

$$(a.) \nabla f = \langle 2x + 4y, 4x + 12y \rangle \therefore (\nabla f)(2, 0) = \langle 4, 8 \rangle$$

$$\text{Observe } \|\nabla f(2, 0)\| = \sqrt{16 + 64} = \sqrt{80} < \sqrt{81} = 9$$

$$\text{Thus } D_{\hat{u}} f(2, 0) = \nabla f(2, 0) \cdot \hat{u} = \sqrt{80} \cos \theta \neq 9 \text{ for any } \theta.$$

∴ there is no direction in which f changes at rate 9 at $(2, 0)$.

$$(b.) \text{ To obtain } D_{\hat{u}} f(2, 0) = 0 = (\nabla f)(2, 0) \cdot \langle a, b \rangle$$

we need $\langle a, b \rangle \perp \langle 4, 8 \rangle$ thus use the vector $\langle -8, 4 \rangle$ normalized,

$$\boxed{\hat{u} = \frac{1}{\sqrt{80}} \langle -8, 4 \rangle \quad \text{or} \quad \hat{u} = \frac{1}{\sqrt{80}} \langle 8, -4 \rangle}$$

$$(4.) (10pts) \text{ Let } f(x, y, z) = z \sin(x^2 + y^2).$$

Calculate the rate of change of f in the $\langle 2, 1, 2 \rangle$ -direction at $\left(\frac{\sqrt{\pi}}{2}, \frac{\sqrt{\pi}}{2}, 17\right) = P$

$$\nabla f = \langle 2xz \cos(x^2 + y^2), 2yz \cos(x^2 + y^2), \sin(x^2 + y^2) \rangle$$

$$\nabla f \left(\frac{\sqrt{\pi}}{2}, \frac{\sqrt{\pi}}{2}, 17 \right) = \langle 17\sqrt{\pi} \cos(\pi/2), 17\sqrt{\pi} \cos(\pi/2), \sin(\pi/2) \rangle$$

$$\underline{\nabla f(P) = \langle 0, 0, 1 \rangle}.$$

Hence, as $\langle 2, 1, 2 \rangle \Rightarrow \hat{u} = \frac{1}{3} \langle 2, 1, 2 \rangle$ since $\|\langle 2, 1, 2 \rangle\| = \sqrt{9}$,

$$(D_{\hat{u}} f)(P) = \langle 0, 0, 1 \rangle \cdot \frac{1}{3} \langle 2, 1, 2 \rangle = \boxed{\frac{2}{3}}$$

(5.) (10pts) Suppose $w = xy + yz$ and $x + y^3 + z = 1$. Calculate $\left(\frac{\partial w}{\partial x}\right)_y$.

$$\begin{aligned} \frac{\partial w}{\partial x} \Big|_y &= \frac{\partial}{\partial x} (xy) + \frac{\partial}{\partial x} (yz) \\ &= y + y \frac{\partial z}{\partial x} \Big|_y \\ &= y + y(-1) \\ &= 0 \end{aligned} \quad \left| \begin{array}{l} x + y^3 + z = 1 \\ 1 + \frac{\partial z}{\partial x} \Big|_y = 0 \\ \therefore \frac{\partial z}{\partial x} \Big|_y = -1 \end{array} \right.$$

Remark: $w = xy + yz = xy + y(1 - x - y^3) = y - y^4$ so our calculation is quite reasonable.

(6.) (10pts) Let $\rho = \sqrt{x^2 + y^2 + z^2}$. Show $\nabla \left(\frac{-1}{\rho} \right) = \frac{1}{\rho^3} \langle x, y, z \rangle$

$$\frac{\partial}{\partial x} \left(\frac{-1}{\rho} \right) = \frac{\partial}{\partial x} \sqrt{x^2 + y^2 + z^2} = \frac{2x}{2\sqrt{x^2 + y^2 + z^2}} = \frac{x}{\rho}$$

$$\text{Likewise } \frac{\partial \rho}{\partial y} = \frac{y}{\rho} \quad \text{and} \quad \frac{\partial \rho}{\partial z} = \frac{z}{\rho}$$

Hence,

$$\begin{aligned} \nabla \left(\frac{-1}{\rho} \right) &= \left\langle \frac{\partial}{\partial x} \left(\frac{-1}{\rho} \right), \frac{\partial}{\partial y} \left(\frac{-1}{\rho} \right), \frac{\partial}{\partial z} \left(\frac{-1}{\rho} \right) \right\rangle \\ &= \left\langle \frac{1}{\rho^2} \frac{\partial \rho}{\partial x}, \frac{1}{\rho^2} \frac{\partial \rho}{\partial y}, \frac{1}{\rho^2} \frac{\partial \rho}{\partial z} \right\rangle \\ &= \left\langle \frac{1}{\rho^2} \frac{x}{\rho}, \frac{1}{\rho^2} \frac{y}{\rho}, \frac{1}{\rho^2} \frac{z}{\rho} \right\rangle \\ &= \underbrace{\frac{1}{\rho^3} \langle x, y, z \rangle}_{\cdot//} \end{aligned}$$

(7.) (10pts) Consider the graph $z = \underbrace{3 + x^2 + y^2}_{f(x,y)}$. Find the equation of the tangent plane at $(-1, 2, 8)$.

$$f_x = 2x \quad \therefore f_x(-1, 2) = 2(-1) = -2.$$

$$f_y = 2y \quad \therefore f_y(-1, 2) = 2(2) = 4.$$

Thus, as $Z = f(-1, 2) + f_x(-1, 2)(x+1) + f_y(-1, 2)(y-2)$ is tangent plane eqⁿ we find,

$$Z = 8 - 2(x+1) + 4(y-2)$$

(8.) (15pts) Consider the surface M given as the solution set of $y^2 + z^2/4 = 1$. Find the following:

(a.) the equation of the tangent plane to M at $(7, 0, 2)$

(b.) a parametrization of the tangent plane to M at $(7, 0, 2)$

(a.) $F(x, y, z) = y^2 + \frac{1}{4}z^2 = 1$ (M is level surface of F)

$$\nabla F = \langle 0, 2y, \frac{1}{2}z \rangle$$

$\nabla F(7, 0, 2) = \langle 0, 0, 1 \rangle$ normal to tangent plane.

$$\Rightarrow 0(x-7) + 0(y-0) + 1(z-2) = 0$$

$$\therefore Z = 2 \text{ eq}^n \text{ of tangent plane.}$$

(b.) Well, (a.) was nicer than I wanted... geometrically we find:

~~param.~~ $\vec{r}(x, y) = (x, y, 2)$

Alternatively, (and this was my intention)

$$x = u, \quad y = \cos\beta, \quad z = 2\sin\beta$$

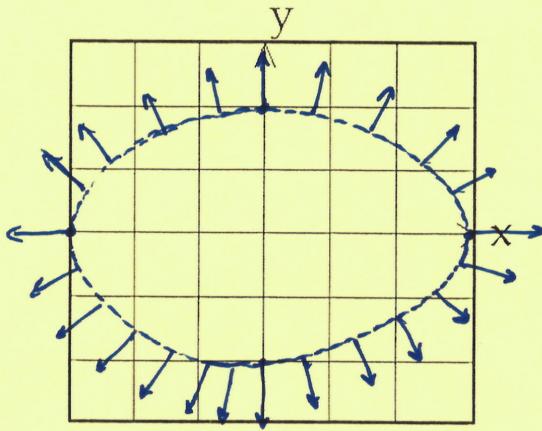
$$\vec{r}(u, \beta) = (u, \cos\beta, 2\sin\beta) = (7, 0, 2) \text{ for what } u \& \beta? \\ \text{clearly, } u = 7, \beta = \pi/2.$$

$$\frac{\partial \vec{r}}{\partial u} = \langle 1, 0, 0 \rangle$$

$$\frac{\partial \vec{r}}{\partial \beta} = \langle 0, -\sin\beta, 2\cos\beta \rangle$$

$\Rightarrow \langle 1, 0, 0 \rangle, \langle 0, -1, 0 \rangle$ tangents
 $\Rightarrow \vec{r}(s, t) = (7, 0, 2) + s\langle 1, 0, 0 \rangle + t\langle 0, -1, 0 \rangle$

- (9.) (5pts) Plot the level curve $F(x, y) = x^2/9 + y^2/4 = 1$ and sketch the gradient vector field ∇F along the curve. The curve plot should use a scale in which each box has length one. The vector field length for ∇F is unimportant, but, the direction is critical.



$$\underline{x=0} \quad 0 + \frac{y^2}{4} = 1 \rightarrow y = \pm 2.$$

$$\underline{y=0} \quad \frac{x^2}{9} + 0 = 1 \rightarrow x = \pm 3$$

$\nabla F \perp$ tangent to curve.

Remember, $F(\vec{r}(t)) = 1$

$$\Rightarrow \underline{\nabla F(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} = 0}.$$

- (10.) (12pts) Let $\vec{r}(s, t) = \langle s^2, t, st \rangle$ be the parametrization of a surface M . Calculate the normal vector field $\vec{N}(s, t) = \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}$ and find the equation of the tangent plane to M at $(4, -1, 2)$.

$$\frac{\partial \vec{r}}{\partial s} = \langle 2s, 0, t \rangle$$

$$\frac{\partial \vec{r}}{\partial t} = \langle 0, 1, s \rangle$$

$$\vec{N}(s, t) = \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} = \boxed{\langle -t, -2s^2, 2s \rangle} \rightsquigarrow \vec{N}(s, t).$$

$$\vec{r}(s, t) = \langle s^2, t, st \rangle = (4, -1, 2) \Rightarrow \underline{t = -1}.$$

$$st = 2 \therefore s(-1) = 2$$

$$\vec{N}(-2, -1) = \langle 1, -8, -4 \rangle \Rightarrow \underline{s = -2}.$$

$$\therefore \frac{x-4 - 8(y+1) - 4(z-2) = 0}{x - 8y - 4z = 4} \text{ or, same answer.}$$

- (11.) (3pts) What is the Cartesian equation of M from the last problem?

$$x = s^2$$

$$y = t \Rightarrow t^2 = y^2$$

$$z = st \Rightarrow z^2 = s^2 t^2 = xy^2 \therefore$$

$$\boxed{z^2 = xy^2}$$

$$\nabla(z^2 - xy^2) = \langle -y^2, -2xy, 2z \rangle \therefore \vec{N}(4, -1, 2) = \langle -1, 8, 4 \rangle \text{ yep.}$$

- (12.) (5pts) Suppose $h_x(1, 2) = 10$ and $h_y(1, 2) = 3$. In addition, $x(u, v) = e^{3u} + \sin(\pi v)$ and $y(u, v) = uv^2 + v$. Let $g(u, v) = h(x(u, v), y(u, v))$ and calculate $\frac{\partial g}{\partial u}$ for $u = 0$ and $v = 2$.

$$\begin{aligned}\frac{\partial g}{\partial u} &= \frac{\partial}{\partial u} \left[h(x(u, v), y(u, v)) \right] = \frac{\partial h}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial h}{\partial y} \frac{\partial y}{\partial u} \quad \left| \begin{array}{l} \frac{\partial x}{\partial u} = 3e^{3u} \\ \frac{\partial y}{\partial u} = v^2 \end{array} \right. \\ \frac{\partial g}{\partial u}(0, 2) &= h_x(1, 2) \frac{\partial x}{\partial u}(0, 2) + h_y(1, 2) \frac{\partial y}{\partial u}(0, 2) \\ &= (10)(3) + (3)(4) \\ &= \boxed{42}.\end{aligned}$$

- (13.) (10pts) A giant has an ice cream cone with a cap of ice cream given by $4x^2 + 5y^2 + 7z^2 = 16$ for $z \geq 0$ (in feet). Suppose a ninja dog named Earl runs over the ice cream and at the point $(1, 1, 1)$ we measure both the dx/dt and dy/dt to be 3 ft/s . What is the speed of Earl at $(1, 1, 1)$?

If x, y, z are funcs. of t with velocities $\dot{x}, \dot{y}, \dot{z}$ and
 $4x^2 + 5y^2 + 7z^2 = 16$ then the chain-rule yields

$$8x\dot{x} + 10y\dot{y} + 14z\dot{z} = 0$$

$$\therefore \dot{z} = \frac{-8x\dot{x} - 10y\dot{y}}{14z}$$

But, $\dot{x} = 3, \dot{y} = 3$ and hence at $(1, 1, 1)$ we find

$$\dot{z} = \frac{-8(3) - 10(3)}{14} = -\frac{54}{14} \quad (\text{ft/s})$$

$$\therefore V = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \quad (\text{speed})$$

$$= \sqrt{3^2 + 3^2 + \left(\frac{54}{14}\right)^2} \quad \text{ft/s}$$

$$= \boxed{\frac{3}{7}\sqrt{179} \text{ ft/s}} \quad \text{aka} \quad \boxed{V = 5.734 \text{ ft/s}}$$