Please work the problems in the white space provided and clearly box your solutions.

Problem 1 (15pts) Calculate the integral below:

$$\int_0^1 \int_1^x \int_{e^y}^{x+y} 2z\,dz\,dy\,dx = \int_0^1 \int_1^x \left[(x+y)^2 - (e^y)^2\right] dy\,dx$$

$$= \int_0^1 \int_1^x \left(x^2 + 2xy + y^2 - e^{2y}\right) dy\,dx$$

$$= \int_0^1 x^2(x-1) + x\left[x^2-1\right] + \tfrac{1}{3}(x^3-1) - \tfrac{1}{2}\left(e^{2x} - e^2\right)\Big] dx$$

$$= \int_0^1 \left[\left(2+\tfrac{1}{3}\right)x^3 - x^2 - x - \tfrac{1}{3} + \tfrac{e^2}{2} - \tfrac{1}{2}e^{2x}\right] dx$$

$$= \frac{7}{3}\frac{1}{4} - \frac{1}{3} - \frac{1}{2} - \frac{1}{3} + \frac{e^2}{2} - \frac{1}{4}\left(e^2 - 1\right)$$

$$= \boxed{\frac{e^2}{4} - \frac{1}{3}}$$

Problem 2 (15pts) Calculate $\frac{\partial(x,y,z)}{\partial(\alpha,\beta,\gamma)}$ given that $x = e^{\alpha+\beta}$ and $y = e^{\beta-\alpha}$ and $z = \sin^2(\gamma^2)$. Simplify your answer.

$$\frac{\partial(x,y,z)}{\partial(\alpha,\beta,\gamma)} = \det\begin{bmatrix} x_\alpha & x_\beta & x_\gamma \\ y_\alpha & y_\beta & y_\gamma \\ z_\alpha & z_\beta & z_\gamma \end{bmatrix} = \det\begin{bmatrix} e^{\alpha+\beta} & e^{\alpha+\beta} & 0 \\ -e^{\beta-\alpha} & e^{\beta-\alpha} & 0 \\ 0 & 0 & 2\gamma\sin(\gamma^2)\cos(\gamma^2) \end{bmatrix}$$

$$= \left(e^{\alpha+\beta}e^{\beta-\alpha} + e^{\alpha+\beta}e^{\beta-\alpha}\right)\left(4\gamma\sin(\gamma^2)\cos(\gamma^2)\right)$$

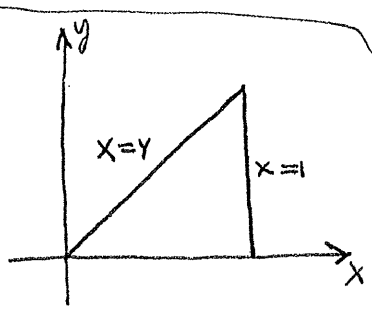
$$= \boxed{4e^{2\beta}\gamma\sin(2\gamma^2)}$$

Problem 3 (15pts) Calculate $\int_0^1 \int_y^1 e^{x^2}\,dx\,dy = \int_0^1 \int_0^x e^{x^2}\,dy\,dx$

$$= \int_0^1 x e^{x^2}\,dx$$

$$= \tfrac{1}{2}e^{x^2}\Big|_0^1$$

$$= \boxed{\tfrac{1}{2}(e-1)}$$



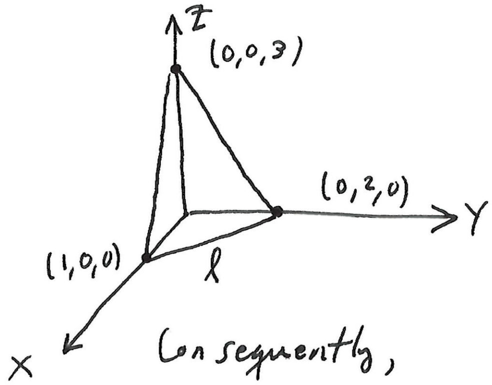
$0 \le y \le 1$ aka $0 \le x \le 1$

$y \le x \le 1$ $0 \le y \le x$

TYPE II $\longrightarrow$ TYPE I

$$z = 3 - 3x - \frac{3}{2}y$$

Problem 4 (15pts) Find the volume bounded by $x + y/2 + z/3 = 1$ and $x \geq 0$, $y \geq 0$ and $z \geq 0$.



$x = y = 0 \Rightarrow z/3 = 1 \quad \therefore (0,0,3)$ on z-axis.
$x = z = 0 \Rightarrow y/2 = 1 \quad \therefore (0,2,0)$ on y-axis
$y = z = 0 \Rightarrow x = 1 \quad \therefore (1,0,0)$ on x-axis.

These points help me draw $\mathcal{P}$

$z = 0 \Rightarrow x + y/2 = 1 \hookrightarrow y = 2 - 2x \ : \ \ell$
for $0 \leq x \leq 1$

Consequently,

$$V = \int_0^1 \int_0^{2-2x} \int_0^{3-3x-\frac{3}{2}y} dz\, dy\, dx = \int_0^1 \int_0^{2-2x} \left(3 - 3x - \frac{3}{2}y\right) dy\, dx$$

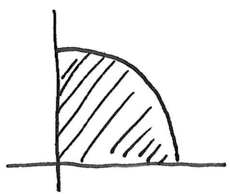
$$= \int_0^1 \left[(3-3x)(2-2x) - \frac{3}{4}(2-2x)^2\right] dx$$

$$= \int_0^1 \left(6 - 12x + 6x^2 - \frac{3}{4}\left[4 - 8x + 4x^2\right]\right) dx = 6 - 6 + \frac{6}{3} - \frac{3}{4}\left[4 - 4 + \frac{4}{3}\right]$$

$$= 2 - 1$$

$$= \boxed{1}$$

Problem 5 (15pts) Let $f(x,y) = xy$ and let D be the quarter disk of radius R in the quadrant where $x \geq 0$ and $y \geq 0$. Calculate $\iint_D f\, dA$. $\quad xy = (r\cos\theta)(r\sin\theta)$, $dA = r\,dr\,d\theta$



$0 \leq \theta \leq \pi/2$
$0 \leq r \leq R$

$$\iint_D f\,dA = \int_0^{\pi/2} \int_0^R r^3 \sin\theta \cos\theta \, d\theta$$

$$= \left(\frac{R^4}{4}\right) \int_0^{\pi/2} \frac{1}{2}\sin(2\theta)\, d\theta$$

$$= \left(\frac{R^4}{4}\right)\left(\frac{-1}{4}\cos(2\theta)\Big|_0^{\pi/2}\right)$$

$$= \frac{R^4}{4}\left(-\frac{1}{4}\cos(\pi) + \frac{1}{4}\cos(0)\right)$$

$$= \boxed{R^4/8}$$

$x^2+y^2=r^2=\rho^2\sin^2\phi$, $dV=\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

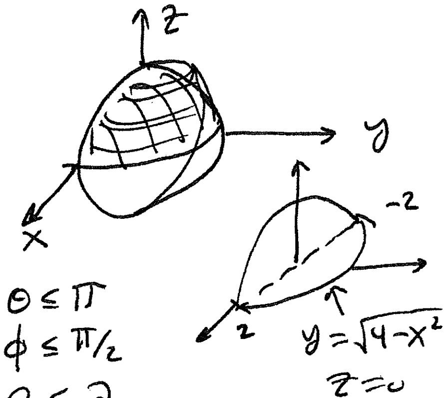
Problem 6 (15pts) Calculate the integral below by changing to spherical coordinates:

$$I = \int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} (x^2+y^2)\, dz\, dy\, dx$$

$-2 \le x \le 2$

$0 \le y \le \sqrt{4-x^2}$

$0 \le z \le \sqrt{4-x^2-y^2}$



$0 \le \theta \le \pi$

$0 \le \phi \le \pi/2$

$0 \le \rho \le 2$

$$I = \int_0^2\int_0^{\pi}\int_0^{\pi/2} \rho^4 \sin^3\phi \, d\phi\, d\theta\, d\rho$$

$$= \left(\frac{\rho^5}{5}\Big|_0^2\right)\left(\theta\Big|_0^{\pi}\right)\int_0^{\pi/2}(1-\cos^2\phi)\sin\phi\, d\phi$$

$u = \cos\phi$

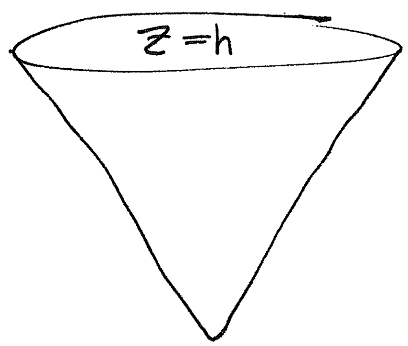
$du = -\sin\phi\, d\phi$

$u(0)=1, \; u(\pi)=-1$

$$= \frac{32\pi}{5}\int_1^{-1}(1-u^2)(-du)$$

$$= \frac{32\pi}{5}\left(\frac{u^3}{3}-u\right)\Bigg|_1^{-1} = \frac{32\pi}{5}\left(\frac{-1}{3}+1-\frac{1}{3}+1\right) = \boxed{\frac{128\pi}{15}}$$

Problem 7 (10pts) Find the volume of the solid bounded by $\underbrace{x^2+y^2}$ $= 3z^2$ and $z = h$ where $h > 0$.



$r^2 = 3z^2$

$r = z\sqrt{3}$

$0 \le \theta \le 2\pi$

$0 \le r \le z\sqrt{3}$

$0 \le z \le h$

$$V = \int_0^{2\pi}\int_0^{h}\int_0^{z\sqrt{3}} r\, dr\, dz\, d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{h}\frac{1}{2}(z\sqrt{3})^2\, dz\right)$$

$$= (2\pi)\left(\frac{3}{2}\frac{z^3}{3}\Big|_0^h\right)$$

$$= \boxed{\pi h^3}$$

$0 \le \rho \le R, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi/2$

Problem 8 (10pts) Let B be the hemisphere formed by $x^2 + y^2 + z^2 \le R^2$ and $z \ge 0$. If the mass density $\delta = \frac{dm}{dV} = k\rho^2$ and B has a total mass of M then find the moment of inertia for B about the z-axis (recall we said $I = \iiint_B (x^2 + y^2)dm$). Do not use k in your answer, instead present the result in terms of M and R.

$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$dm = k\rho^2 dV$

$$M = \iiint_B dm = \int_0^{2\pi}\int_0^{\pi/2}\int_0^R k\rho^4 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= k\left(\frac{R^5}{5}\right)(2\pi)\left(-\cos\phi\right)\Big|_0^{\pi/2}$$

$$= \frac{2\pi k R^5}{5} \quad \therefore \quad k = \frac{5M}{2\pi R^5}$$

$$I = \iiint_B (x^2+y^2)\,dm$$

$$= \int_0^R \int_0^{2\pi} \int_0^{\pi/2} (\rho^2 \sin^2\phi)(k\rho^2)(\rho^2 \sin\phi \, d\phi \, d\theta \, d\rho)$$

$$= \left(\int_0^R \rho^6 d\rho\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi/2} k \sin^3\phi \, d\phi\right)$$

$u = \cos\phi$

$$= \left(\frac{R^7}{7}\right)(2\pi)\left(\frac{4k}{3}\right) \quad \text{as} \quad \left(\int_0^{\pi/2} \sin^3\phi \, d\phi = \int_{-1}^{1} (u^2-1)\,du = \frac{4}{3}\right)$$

(See PROBLEM 6 OR CALCULUS II)

$$= \left(\frac{8\pi R^7}{21}\right)(k)$$

$$= \left(\frac{8\pi R^7}{21}\right)\left(\frac{5M}{2\pi R^5}\right)$$

$$= \boxed{\frac{20}{21} M R^2}$$