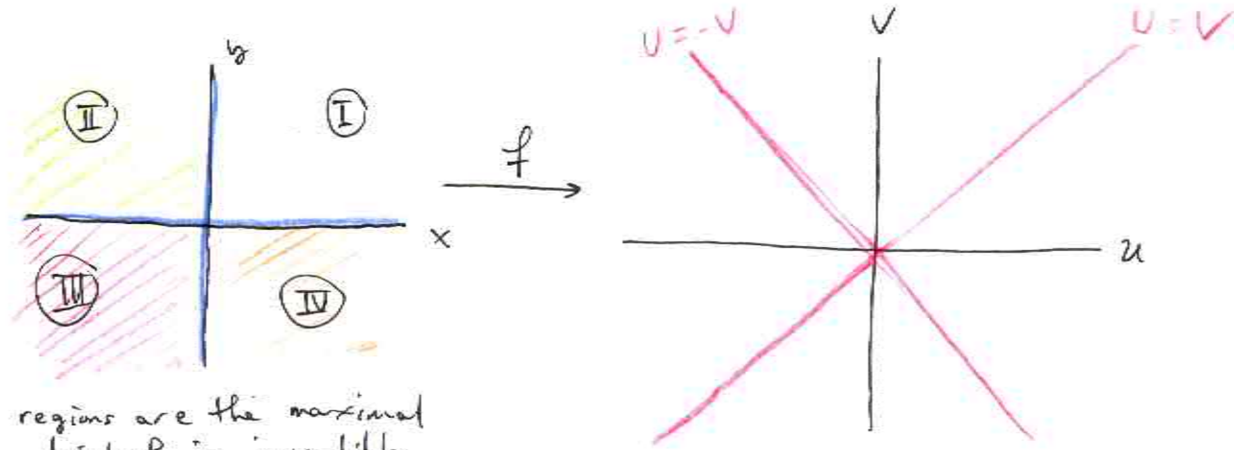
1/10/2001



These 4 regions are the maximal sets on which f is invertible.

$$f_{\mathrm{I}}^{-1}(u,v) = \left(\sqrt{\frac{u+v}{2}}\,,\ \sqrt{\frac{u-v}{2}}\right)$$

$$f_{\mathrm{III}}^{-1}(u,v) = \left(-\sqrt{\frac{u+v}{2}}\,,\ -\sqrt{\frac{u-v}{2}}\right)$$

FACTS: Let $f : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ with $Df(x_0)(h) = h\, J_f(x_0)^T$

$[Df(x_0)$ is invertible$]$ iff $[\, Df(x_0)h = 0 \Rightarrow h = 0\,]$

which is true iff $[\, h\, J_f(x_0)^t = 0 \Rightarrow h = 0\,]$

which is true iff $[$nullspace of $J_f(x_0) = \{0\}]$

then we recall $n = \text{rank } J_f(x_0) + \dim(\text{Nullspace } J_f(x_0))$

$\therefore \quad n = rk(J_f(x_0)) \iff J_f(x_0)$ has inverse

Theorem : Implicit Function Theorem

Assume $U \subseteq \mathbb{R}^{n+m}$ is open and that $F : U \to \mathbb{R}^m$ is smooth. If $(a,b) \in \mathbb{R}^n \times \mathbb{R}^m$ and the following conditions hold

1.) $F(a,b) = 0$

2.) $\det\left(\frac{\partial F^i}{\partial y^j}\right)(a,b) \neq 0 \quad \begin{array}{l} i = 1,2,\dots m \\ j = 1,2,\dots m \end{array}$

Then there exists V_a open about a in $\mathbb{R}^n$ and W_b open about b in $\mathbb{R}^m$ and a smooth mapping (unique) $g : V_a \to W_b$ such that

1.) $g(A) = b$

2.) for $(x,y) \in V_a \times W_b$, $F(x,y) = 0 \iff y = g(x)$

Corollary $\to$ $F(x, g(x) = 0 \Rightarrow \frac{\partial F}{\partial x^i}(x, g(x)) + \sum_{j=1}^{n} \frac{\partial F^i}{\partial y^j} \boxed{\frac{\partial g^j}{\partial x^i}} = 0$

Example

$$x^2 + yz + z^3 - xu^2 - yv^2 = 0$$
$$xyz - yu^2 + xv^2 = 0$$

$$F(x,y,z,u,v) = (x^2 + yz + z^3 - xu^2 - yv^2,\ xyz - yu^2 + xv^2)$$
$$F : \mathbb{R}^5 \to \mathbb{R}^2$$

Choose $(a,b) = ((1,1,1),(b_1,b_2))$ so that $F(a,b) = 0$

$$F(a, b_1, b_2) = (3 - b_1^2 - b_2^2,\ 1 - b_1^2 + b_2^2) \Rightarrow \begin{array}{l} b_1^2 + b_2^2 = 3 \\ b_1^2 - b_2^2 = 1 \end{array} \Rightarrow \begin{array}{l} b_1 = \pm\sqrt{2} \\ b_2 = \pm 1 \end{array}$$

So we choose $b = (\sqrt{2}, 1)$ thus

$$F(1,1,1,\sqrt{2},1) = 0$$

$$F^1(x,y,z,u,v) = x^2 + yz - xu^2 - yv^2$$
$$F^2(x,y,z,u,v) = xyz - yu^2 + xv^2$$

$$\left(\frac{\partial F^i}{\partial y^j}\right) = \frac{\partial(F^1, F^2)}{\partial(u,v)} = \begin{pmatrix} \frac{\partial F^1}{\partial u} & \frac{\partial F^1}{\partial v} \\ \frac{\partial F^2}{\partial u} & \frac{\partial F^2}{\partial v} \end{pmatrix} = \begin{pmatrix} -2xu & -2yv \\ -2yu & 2xv \end{pmatrix}$$

$$\det\left(\frac{\partial F^i}{\partial y^j}\right) = -4x^2uv - 4y^2uv = (-4uv)(x^2+y^2) \neq 0 \quad \text{at } (1,1,1,\sqrt{2},1)$$

So we know the function like g exists and here we can even find it

$$xu^2 + yv^2 = x^2 + yz + z^3 \equiv f_1(x,y,z)$$
$$yu^2 - xv^2 = xyz \equiv f_2(x,y,z)$$

Multiplying by x and y yields

$$(x^2+y^2)u^2 = xf_1 + yf_2$$
$$u^2 = \frac{xf_1 + yf_2}{x^2+y^2}$$

So we have arrived at

$$(u,v) = g(x,y,z)$$
$$u = g_1(x,y,z)$$
$$v = g_2(x,y,z)$$
$$g(1,1,1) = (\sqrt{2}, 1) \quad \leftarrow g(a) = b$$

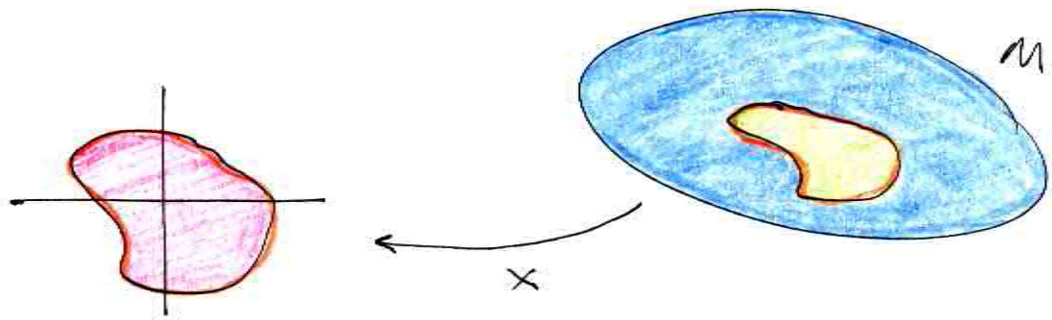
Thus we choose the positive root

$$u = \sqrt{\frac{xf_1 + yf_2}{x^2+y^2}}$$

Def$^{\underline{n}}$/ If M is a set then (U, x) is a chart on M iff $U \subseteq M$ and $x(U)$ is an open subset of $\mathbb{R}^m$ for some m and the following map x,

$$x : U \longrightarrow x(U) \subseteq M$$

is a bijection. Then $x(U) = \{x(p) \mid p \in U\}$



Def$^{\underline{n}}$/ Two charts (U, x) and (V, y) on M are compatible iff either $U \cap V = \phi$ or $x(U \cap V)$ and $y(U, V)$ are open and $y \circ x^{-1} : x(U \cap V) \rightarrow y(U \cap V)$ is a diffeomorphism

$$D\varphi^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$id_{\mathbb{R}^n} = D(\varphi \circ \varphi^{-1}) = D\varphi \circ D\varphi^{-1} \quad \therefore \quad m \geq n$$

$$id_{\mathbb{R}^m} = D(\varphi^{-1} \circ \varphi) = D\varphi^{-1} \circ D\varphi \quad \therefore \quad n \geq m$$

$$\therefore \quad m = n$$

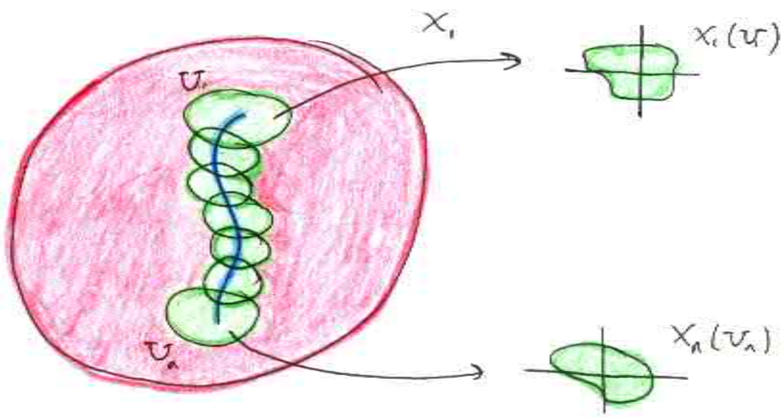
Comment on Differentiation

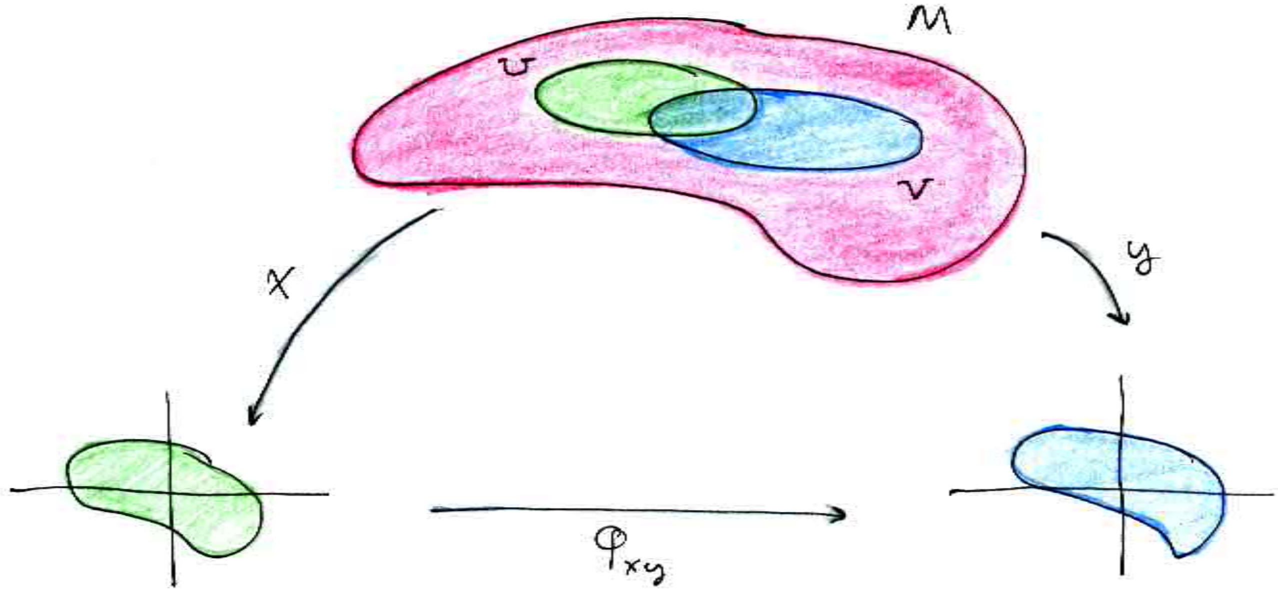
$$f : U \subseteq \mathbb{R}^k \longrightarrow V \subseteq \mathbb{R}^l$$
$$g : V \subseteq \mathbb{R}^l \longrightarrow \mathbb{R}^q$$

$$D(g \circ f) = (Dg) \circ (Df)$$

$$J_{g \circ f} = J_g J_f$$

Dimensionality of a Manifold





$$\left(y^1(p),\, y^2(p), \ldots, y^m(p) \right) = \varphi_{xy}\left(x^1(p),\, x^2(p), \ldots, x^m(p) \right)$$

$$\varphi_{xy}(u^1, u^2, \ldots, u^m) = (v^1, v^2, \ldots, v^m)$$

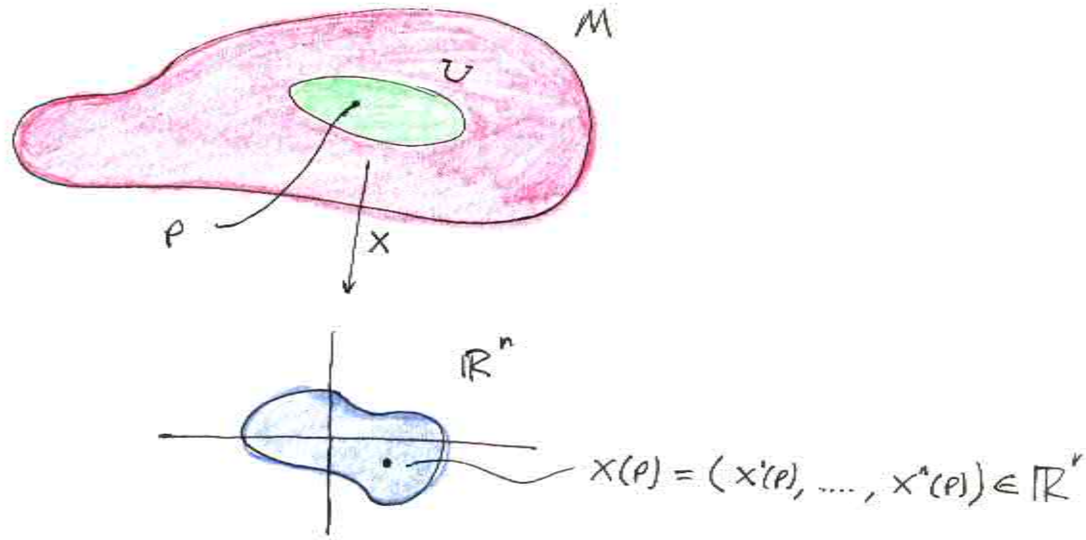
$$\frac{\partial \varphi^i_{xy}}{\partial u^j}\big(x(q)\big) := \frac{\partial y^i}{\partial x^j}(q) \qquad (\text{LHS def}^{\underline{n}}\text{'s RHS})$$

$$\frac{\partial y^i}{\partial x^j} = \frac{\partial}{\partial u^j}\left(y^i \circ x^{-1} \right) \qquad \frac{\partial y^i}{\partial x^j} : U \cap V \longrightarrow \mathbb{R} \quad !$$

$$J_{\varphi_{xy}} = \frac{\partial \varphi^i_{xy}}{\partial u^j}$$

1/18/2001

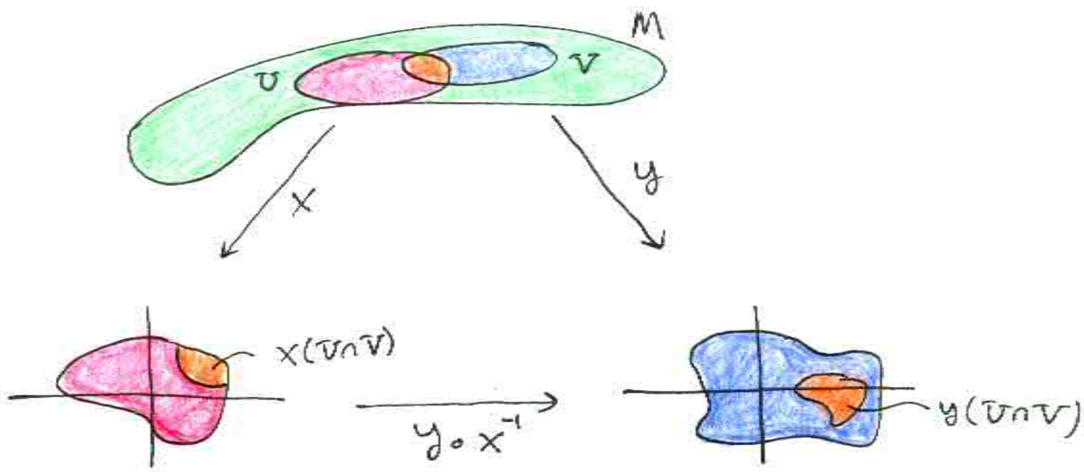
> **Def^{n}/** Let M be a set. A chart on M is a pair (U, x) such that $U \subseteq M$ and $x : U \to$ is a bijection from $U \to x(U)$ where $x(U) = \{x(p) \mid p \in U\} \subseteq \mathbb{R}^m$ and is open.



Def^{n}/ If (U, x) and (V, y) are charts on M then we say that these charts are <u>compatible</u> iff either $U \cap V = \emptyset$ or

1.) $x(U \cap V)$ is open in $x(U)$
$\quad y(U \cap V)$ is open in $y(V)$

2.) The mapping $y \circ x^{-1} : x(U \cap V) \to y(U \cap V)$ and also $x \circ y^{-1} : y(U \cap V) \to x(U \cap V)$ are smooth.



Lemma: If $L : \mathbb{R}^m \to \mathbb{R}^n$ is a linear injection then $m \le n$.

Pf: Let $e_1, e_2, \ldots, e_m$ be the standard basis of $\mathbb{R}^m$

Further let $f_i = L(e_i)$ we claim $f_1, \ldots, f_m$ are linearly independent, $\sum_{i=m}^{m} \lambda_i f_i = 0 \Rightarrow \sum_{i=1}^{m} \lambda_i L(e_i) = 0$

$$\Rightarrow L\left(\sum \lambda_i e_i\right) = 0 = L(0) \Rightarrow \sum \lambda_i e_i = 0$$

$$\Rightarrow (\lambda_1, \ldots, \lambda_m) = (0, 0, \ldots, 0)$$

So then $f_1, f_2, \ldots f_m \in \mathbb{R}^n$ then we can extend $f_1, f_2, \ldots, f_m, f_{m+1}, \ldots f_n$ such that this is a basis of $\mathbb{R}^n$

$\therefore m \le n$

Lemma: If $L : \mathbb{R}^m \to \mathbb{R}^n$ and $S : \mathbb{R}^n \to \mathbb{R}^m$ are linear and $S \circ L = id_{\mathbb{R}^m}$ then L is injective

Pf: Let $x, y \in \mathbb{R}^m$ such that $L(x) = L(y)$

Then $S(L(x)) = S(L(y)) \therefore x = y$

Proposition: Assume M is a set, that (U, x) and (V, y) are charts on M which are compatible. If $x(U) \subseteq \mathbb{R}^m$ and $y(V) \subseteq \mathbb{R}^n$ and if $U \cap V \neq \phi$ then $n = m$

Pf: Let $\varphi = \varphi_{xy} = y \circ x^{-1}$

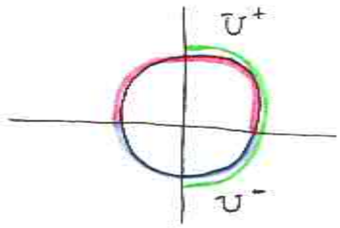
$$\varphi_{xy} : x(U \cap V) \subseteq \mathbb{R}^m \longrightarrow y(U \cap V) \subseteq \mathbb{R}^n$$

$$D\varphi = D\varphi_{xy} : \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

? isn't clear from φ_{xy} & φ_{xy}^{-1} that $x(U \cap V) \cong y(U \cap V)$

Def$^{\underline{n}}$/ If M is a set we say that a collection $\mathcal{A}$ of charts on M is an atlas iff every pair of charts in $\mathcal{A}$ is compatible and if every point of M is in the domain of some chart in $\mathcal{A}$

Example



$$x^+ : U^+ \longrightarrow \mathbb{R} \qquad x^+(p_1,p_2) = p_1$$
$$x^- : U^- \longrightarrow \mathbb{R} \qquad x^-(p_1,p_2) = p_1$$

$$U^+ = \{(p_1,p_2) \mid p_1^2+p_2^2 = 1 \,,\ p_2 > 0\}$$
$$x^+(U^+) = (-1,1) \subseteq \mathbb{R}$$
$$U^- = \{(p_1,p_2) \mid p_1^2+p_2^2 = 1 \,,\ p_2 < 0\}$$
$$x^-(U^-) = (-1,1) \subseteq \mathbb{R}$$
$$V^+ = \{(p_1,p_2) \mid p_1 > 0\} \qquad y^+(p_1,p_2) = p_2$$

$$\varphi = y^+ \circ (x^+)^{-1} \quad ; \quad \varphi : (0,1) \longrightarrow (0,1)$$
$$\varphi(t) = y^+\left((x^+)^{-1}(t)\right) = y^+\left(t, \sqrt{1-t^2}\right) = \sqrt{1-t^2}$$
$\therefore$ φ is smooth on $(0,1)$

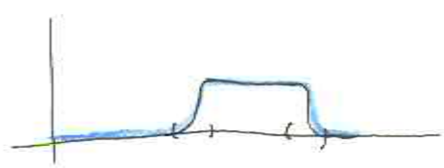
Homework

Spherical Coordinates $(\rho, \theta, \varphi) \longrightarrow (x, y, z)$
$$x = \rho\cos\theta\sin\varphi$$
$$y = \rho\sin\theta\sin\varphi$$
$$z = \rho\cos\varphi$$

Due Friday:

Excersise 4 : Don't compute ~~potentials~~ partials

Ex 1.1.1
1.2.1
1.2.2



1.1.1

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x} = 0$$

$$f(x) = \begin{cases} 0 & x \le 0 \\ e^{-1/x} & x > 0 \end{cases}$$

example of function that is C^∞ on whole $\mathbb{R}$ but is not analytic.

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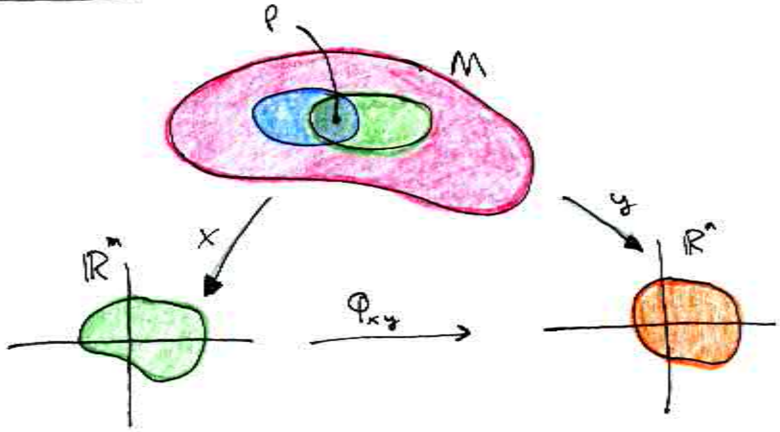
Topological Manifolds



Brawer invariance of domain theorem

φ_{xy} a homeomorphism, we still have the preservation of dimension among compatible charts. Dimensionality is well defined for a Topological Manifold; which has transition maps which are homeomorphisms.

Theorem

assume $\mathcal{a}$ is an atlas on a set M and then let

$$\mathcal{a}^* = \{ (U, x) \mid (U, x) \text{ is a chart which is compatible with all of } \mathcal{a} \}$$

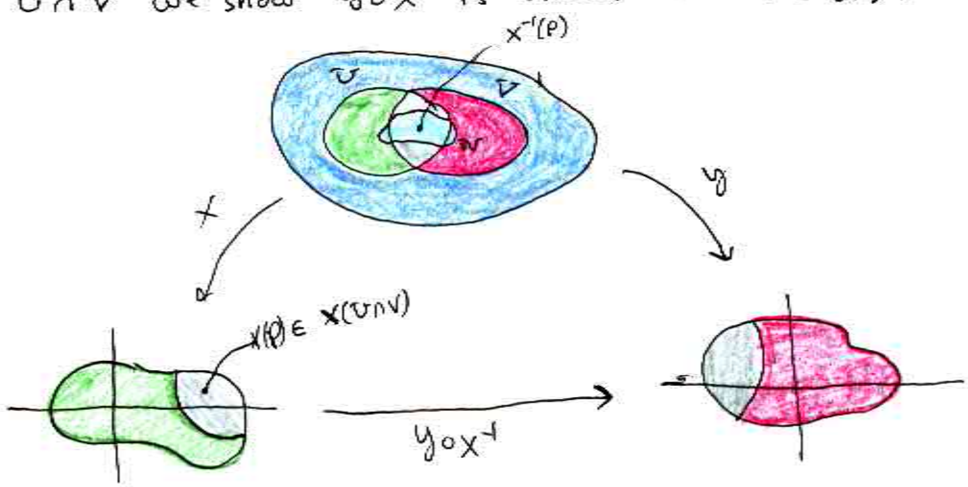
Then we call $\mathcal{a}^*$ the maximal atlas on M containing $\mathcal{a}$.

Every atlas which contains $\mathcal{a}$ is contained in $\mathcal{a}^*$

Pf/ Assume (U, x) and (V, y) are in $\mathcal{a}^*$ and $U \cap V \neq \emptyset$. We show that they are compatible. That is $y \circ x^{-1}$ is smooth.

$$y \circ x^{-1} : x(U \cap V) \to y(U \cap V)$$

Let $p \in U \cap V$ we show $y \circ x^{-1}$ is smooth in nbhd(p).



$(U, x) \in \mathcal{A}^*$

Choose $(W, z) \in \mathcal{A}$ such that $x^{-1}(p) \in W$

$(U, x) \in \mathcal{A}^* \Rightarrow (U, x) \sim (W, z)$

$(V, y) \in \mathcal{A}^* \Rightarrow (V, y) \sim (W, z)$

Consider $W \cap U \cap V$

$$y \circ x^{-1} = y \circ z^{-1} \circ z \circ x^{-1} = (y \circ z^{-1}) \circ (z \circ x^{-1})$$

This all makes sense if we restrict the above to $x(W \cap U \cap V)$

Then this is a map which is the composition of 2 smooth functions $\therefore$ $y \circ x^{-1}$ is smooth on $x(W \cap U \cap V)$

$\therefore$ $y \circ x^{-1}$ is smooth on $x(U \cap V)$

$(U, x) \in \mathcal{A} \Rightarrow (U, x) \in \mathcal{A}^*$ $\therefore$ $\mathcal{A} \subseteq \mathcal{A}^*$ duh.

§ $\mathcal{B}$ is an atlas such that $\mathcal{A} \subseteq \mathcal{B}$ is $\mathcal{B} \subseteq \mathcal{A}^*$?

$(U, x) \in \mathcal{B}$ then (U, x) is compatible with every other chart in $\mathcal{B}$ $\therefore$ with every other chart in $\mathcal{A}$ $\therefore$ $(U, x) \in \mathcal{A}^*$.

$\boxed{\text{QED}}$

Def$^{\underline{n}}$/ MANIFOLD

If M is a set then a maximal atlas on M is called a differentiable structure on M. A set with a differentiable structure is called a manifold.

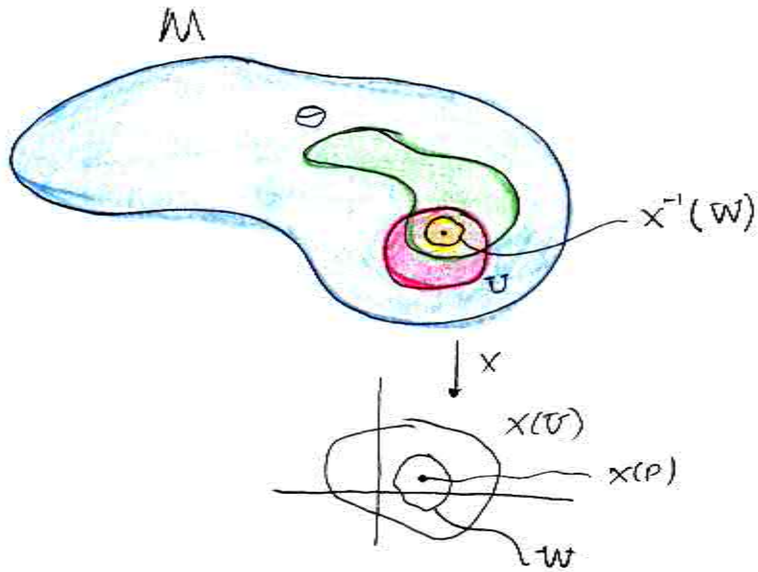
1/23/2001

Def^{n}/ TOPOLOGY ON MANIFOLDS

If M is a manifold then $\Theta \subseteq M$ is open iff $\forall p \in \Theta$ $\exists$ a chart (U, x) such that

(1.) $p \in U$

(2.) $\exists$ open set $W \subseteq x(U)$ such that $x(p) \in W$ and $p \in x^{-1}(W) \subseteq \Theta$



Def^{n}/ Θ is open in M iff $\forall p \in \Theta$ $\exists$ a chart domain V such that $p \in V \subseteq \Theta$.

Example $U \overset{open}{\subseteq} \mathbb{R}^n$

$\mathcal{A} = \{(U, id_U)\} \Rightarrow \mathcal{A}^*$ standard differentiable structure on $\mathbb{R}^n$