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General Stereographic Projection

Def^n/ A n-sphere is a subset $S^n \subseteq \mathbb{R}^{n+1}$ consisting of points $(x^1, x^2, \dots, x^{n+1}) \in \mathbb{R}^{n+1}$ subject to

$$(x^1)^2 + (x^2)^2 + \dots + (x^{n+1})^2 = 1$$

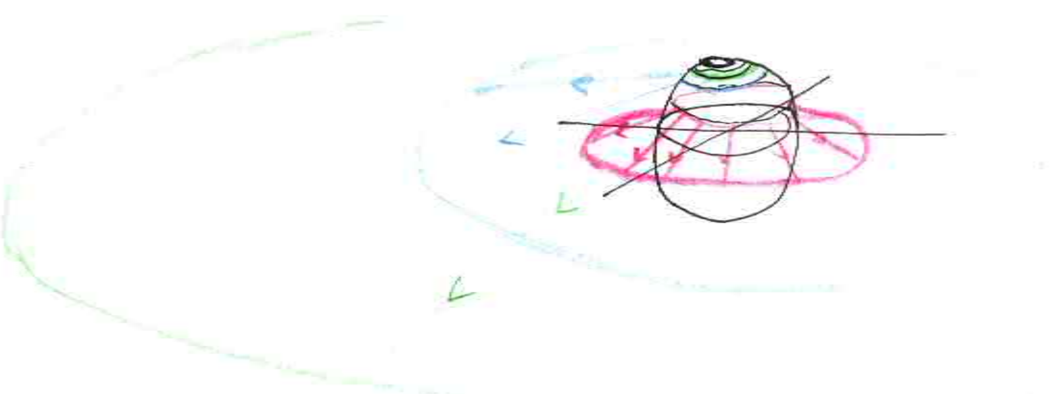
S^4 of use in math physics. In particular $\exists$ a 1-1 correspondance between $S^4 - \{\text{North Pole}\} \longleftrightarrow \mathbb{R}^4$, topologists say S^4 is the one point compactification of $\mathbb{R}^4$...

$$\mathcal{L}\phi \quad , \quad \phi : \mathbb{R}^4 \to \mathbb{C}^4 \text{ or whatever but}$$

many functions have domain $\mathbb{R}^4 \simeq$ Minkowski Space

$$\int_{\mathbb{R}^4} \phi(x)\, d^4x = ? \qquad \text{well we see the}$$

need for $\phi(x) \to 0$ for large x.



Define field on compactified space is way to describe asymptotic behaviour of fields

$$\begin{array}{c} S^3 \\ \pi \big\downarrow \\ S^2 \end{array}$$

$\pi^{-1}(p)$ is a circle imbedded in S^3

can use for monopole description

Stereographic Projection

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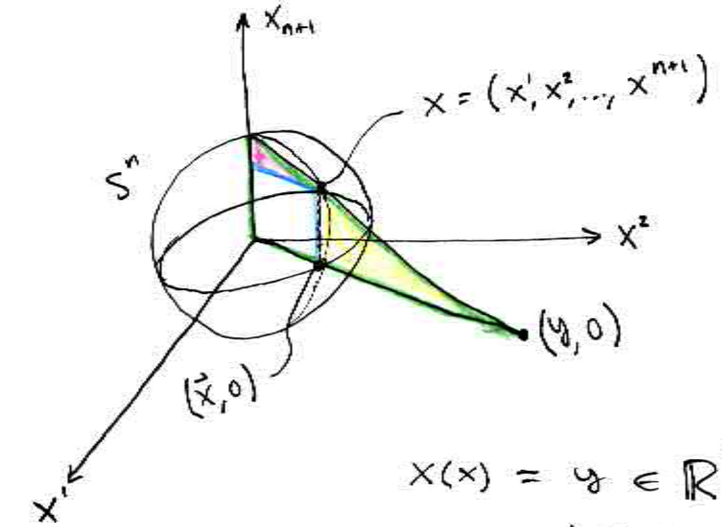


$$\chi(x) = y \in \mathbb{R}^n$$

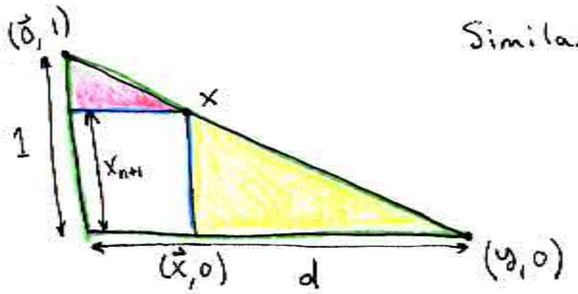
$$x = (x^1, x^2, \dots, x^{n+1})$$

$$y = (y^1, y^2, \dots, y^n)$$

$$\chi : S^n \subseteq \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$$

$$\vec{x} = (x^1, x^2, \dots, x^n)$$

$\dfrac{\vec{x}}{\|\vec{x}\|_n}$ is a unit vector pointing towards $(y, 0)$



Similar Triangles yields the equivalent ratios,

$$\frac{1 - x^{n+1}}{\|\vec{x}\|} = \frac{1}{d}$$

$$\chi(x) = d \, \frac{\vec{x}}{\|\vec{x}\|} = \frac{\|\vec{x}\|}{1 - x^{n+1}} \, \frac{\vec{x}}{\|\vec{x}\|}$$

$$\boxed{\chi(x) = \left(\frac{1}{1 - x^{n+1}}\right)(x^1, x^2, \dots, x^n)}$$

Let $N = (0, 0, \dots, 1) = (\vec{0}, 1)$ then we see that χ is defined on

$$\chi : S^n - \{N\} \longrightarrow \mathbb{R}^n$$

Can we find the inverse on the fly? ; $\chi^{-1}(y) = x$

$$y = \frac{1}{1-x^{n+1}}(x^1, x^2, \dots, x^n)$$

$$y^i = \frac{1}{1-x^{n+1}} x^i$$

$$x^i = (1-x^{n+1})\, y^i$$

$$x^{n+1} = \frac{\|y\|^2 - 1}{\|y\|^2 + 1} \Rightarrow 1 - x^{n+1} = \frac{2}{\|y\|^2 + 1}$$

missing some algebra here.

$$\chi^{-1}(y) = x = (x^1, x^2, \dots, x^{n+1})$$

$$\chi^{-1}(y) = \left(\frac{2y^1}{\|y\|^2+1}, \frac{2y^2}{\|y\|^2+1}, \dots, \frac{2y^n}{\|y\|^2+1}, \frac{\|y\|^2 - 1}{\|y\|^2+1}\right)$$

Likewise we could project from "south pole" and get

$$\chi^-(x) = \left(\frac{1}{1+x^{n+1}}\right)(x^1, x^2, \dots, x^n)$$

$[\chi^- \circ (\chi^+)^{-1}](y)$ is smooth transition function between first map $\chi \to \chi^+$ and the map χ^-

PRODUCT OF TWO MANIFOLDS

Assume $\mathcal{A}$ is an atlas on a set M and $\mathcal{B}$ is an atlas on a set N. Further if $(U, \mu) \in \mathcal{A}$ and $(V, \nu) \in \mathcal{B}$ then define $(U \times V, \mu \times \nu)$ as follows

$$U \times V = \left\{ (a, b) \mid a \in U, b \in V \right\}$$

$$\mu \times \nu \doteq U \times V \longrightarrow \mu(U) \times \nu(V) \subseteq \mathbb{R}^m \times \mathbb{R}^n \equiv \mathbb{R}^{m+n}$$

$$(\mu \times \nu)(u, v) \equiv (\mu(u), \nu(v))$$

$$z = (x, y), \quad x \in \mathbb{R}^m, y \in \mathbb{R}^n$$

$$\|z\|^2_{m+n} = \|x\|^2_m + \|y\|^2_n$$

given that open sets can be defined by open balls in $\mathbb{R}^K$ we can show that x type open set and $V \subseteq \mathbb{R}^n$ open can be used to show $\exists$ a open ball around a set in an open set in $\mathbb{R}^{m+n}$

$$(\mu \times \nu)(u, v) = (\mu(u), \nu(v))$$

$$\left((f_1 \times g_1) \circ (f_2 \times g_2)\right)(u, v) = (f_1 \circ f_2) \times (g_1 \circ g_2)(u, v)$$

Notice that,

$$(x \times y) \circ (x^{-1} \times y^{-1}) = id_{x(U)} \times id_{y(V)} = id_{x(U) \times Y(V)}$$

$$(x^{-1} \times y^{-1}) \circ (x \times y) = id_U \times id_V = id_{U \times V}$$

Thus $x \times y$ is a bijection from $U \times V$ onto $x(U) \times y(V) \underset{open}{\subseteq} \mathbb{R}^{m+n}$

$\therefore$ $(U \times V, x \times y)$ is an chart on $M \times N$

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Define $\mathcal{A} \otimes \mathcal{B} = \{ (U \times V, x \times y) \mid (U, x) \in \mathcal{A}, (V, y) \in \mathcal{B} \}$

We are obliged to show that $\mathcal{A} \otimes \mathcal{B}$ is indeed an atlas

$$(U_1 \times V_1, x_1 \times y_1)$$
$$(U_2 \times V_2, x_2 \times y_2)$$

$$(a, b) \in (U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$$

$U_1 \cap U_2$ has both x_1 and x_2 defined on it

$V_1 \cap V_2$ has both y_1 and y_2 defined on it

$$(x_1 \times y_1)\left[(U_1 \cap U_2) \times (V_1 \cap V_2)\right] = \underset{\text{open}}{\underset{\uparrow}{x_1(U_1 \cap U_2)}} \times \underset{\text{open}}{\underset{\uparrow}{y_1(V_1 \cap V_2)}}$$

$$(x_2 \times y_2) \circ (x_1 \times y_1)^{-1} = (x_2 \times y_2) \circ (x_1^{-1} \times y_1^{-1})$$
$$= \underset{\text{smooth}}{\underset{\uparrow}{(x_2 \circ x_1^{-1})}} \times \underset{\text{smooth}}{\underset{\uparrow}{(y_2 \circ y_1^{-1})}}$$

$\therefore$ $\mathcal{A} \otimes \mathcal{B}$ is an atlas and is $\therefore$ contained in an unique maximal atlas which defines a differentiable structure. We call this the product structure on the product manifold.

PAGE 31 : PATCHING MANIFOLDS OF A VERY SPECIFIC TYPE TOGETHER

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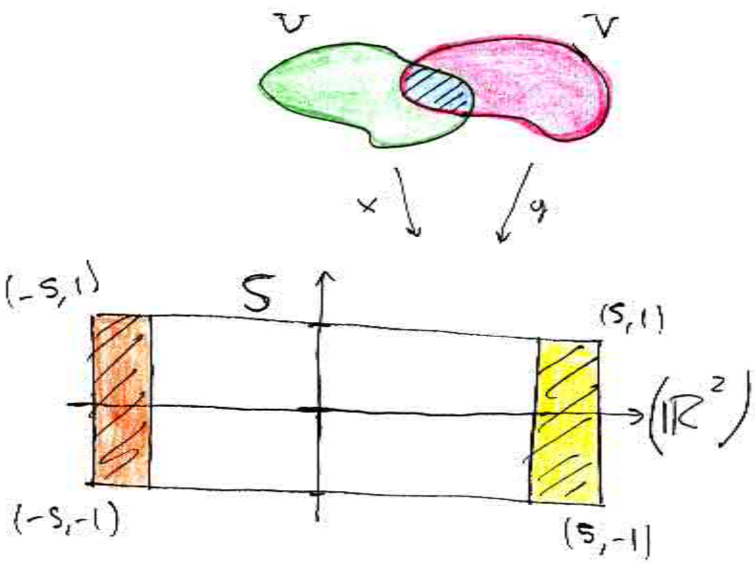
(U, x) chart $\quad x : U \to x(U) \overset{\text{open}}{\subseteq} \mathbb{R}^m$

(V, y) chart $\quad y : V \to y(V) \overset{\text{open}}{\subseteq} \mathbb{R}^m$

Assume $x(U) = y(V) = S$ where $S = (-5,5) \times (-1,1)$, $m = 2$.



$$x(U \cap V) = y(U \cap V) = \mathcal{U}$$

$$\mathcal{U} = -[\underbrace{(-5,-4) \times (-1,1)}_{\mathcal{U}_1}] \cup [\underbrace{(4,5) \times (-1,1)}_{\mathcal{U}_2}] = [\mathcal{U}_1 \cup \mathcal{U}_2]$$

$y \circ x^{-1} : x(U \cap V) \longrightarrow y(U \cap V)$

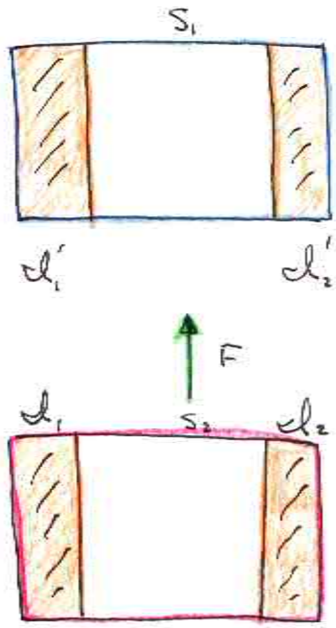
$y \circ x^{-1} : \mathcal{U} \longrightarrow \mathcal{U}$ Which is defined by

$$(y \circ x^{-1})(a,b) = \begin{cases} (a+9, b) & -5 < a < -4 \ , \ -1 < b < 1 \\ (a-9, b) & 4 < a < 5 \ , \ -1 < b < 1 \end{cases}$$

THIS $y \circ x^{-1}$ is clearly smooth and so $(U \cup V)$ is a manifold where $\{(U,x), (V,y)\}$ is the atlas.

A Better Method of Gluing Manifolds

$$S = (-5, 5) \times (-1, 1)$$
$$S_1 = \{(s, 1) \mid s \in S\}$$
$$S_2 = \{(t, 2) \mid t \in S\}$$



$$F : \mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2 \to \mathcal{U}' = \mathcal{U}_1' \cup \mathcal{U}_2'$$

$$F(a,b)_2 = \begin{cases} (a+9,\ b)_1 & a \in (-5,-4) \\ (a-9,\ b)_1 & a \in (4,5) \end{cases}$$

$$G : \mathcal{U} \to \mathcal{U}'$$

$$G(a,b)_2 = \begin{cases} (a+9,\ b)_1 & a \in (-5,-4) \\ (a-9,\ -b)_1 & a \in (4,5) \end{cases}$$

$$(s, 1) \sim (t, 1) \iff s = t$$
$$(s, 2) \sim (t, 2) \iff s = t$$
$$(s, 1) \sim (t, 2) \iff s = F(t)$$
$$(s, 2) \sim (t, 1) \iff t = F(s)$$

Given a point in S_1 containing $(s, 1)$ then $[(s,1)] = [s,1]$ = equivalence class containing $(s,1)$ under $\sim$ equivalence.

$$U_1 = \{[s, 1] \mid s \in S\}$$

$$U_2 = \{[t, 2] \mid t \in S\}$$

Then we have these 1-1 maps x_1 and x_2 such that

$x_1^{-1} : S \to U_1$ with $x_1^{-1}(s) = [s, 1]$

$x_2^{-1} : S \to U_2$ with $x_2^{-1}(t) = [t, 2]$

$x_1 : U_1 \to S$ and $x_2 : U_2 \to S$

Let $P = U_1 \cup U_2$

$$q \in U_1 \cap U_2 \Leftrightarrow q \in U_1 \text{ and } q \in U_2$$
$$\Leftrightarrow q \in [s, 1] \text{ and } q \in [t, 2] \text{ , } s, t \in S$$
$$\Leftrightarrow [s, 1] = [t, 2]$$
$$\Leftrightarrow F(t) = s$$

$$X_1(U_1 \cap U_2) = \mathcal{l}_1' \cup \mathcal{l}_2'$$
$$X_2(U_1 \cap U_2) = \mathcal{l}_1 \cup \mathcal{l}_2$$
(?)

So then we could put G into F arguments just the same,

$$X_1 \circ X_2^{-1} = F$$

The Projective Plane

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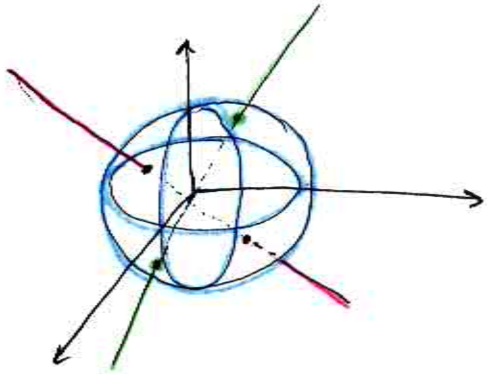
$$P^3\mathbb{R} = \text{set of all lines in } \mathbb{R}^3 \text{ through } (0,0,0)$$



$$U^+ = \{(x,y,z) \in S^2 \mid z > 0\} \qquad \pi_z(x,y,z) = (x,y)$$

$$V^+ = \{(x,y,z) \in S^2 \mid y > 0\} \qquad \pi_y(x,y,z) = (x,z)$$

$$W^+ = \{(x,y,z) \in S^2 \mid x > 0\} \qquad \pi_x(x,y,z) = (y,z)$$

$$\tilde{U}^+ = \{\ell \in P^3\mathbb{R} \mid \ell \cap U^+ \neq \phi\}$$

$$\tilde{V}^+ = \{\ell \in P^3\mathbb{R} \mid \ell \cap V^+ \neq \phi\}$$

$$\tilde{W}^+ = \{\ell \in P^3\mathbb{R} \mid \ell \cap W^+ \neq \phi\}$$

$$\tilde{\pi}_z : \tilde{U}^+ \to \mathbb{R}^2 \quad \text{with} \quad \tilde{\pi}_z(\ell) = \pi_z(\ell \cap U^+)$$

$$\tilde{\pi}_y : \tilde{V}^+ \to \mathbb{R}^2 \quad \text{with} \quad \tilde{\pi}_y(\ell) = \pi_y(\ell \cap V^+)$$

$$\tilde{\pi}_x : \tilde{W}^+ \to \mathbb{R}^2 \quad \text{with} \quad \tilde{\pi}_z(\ell) = \pi_z(\ell \cap W^+)$$

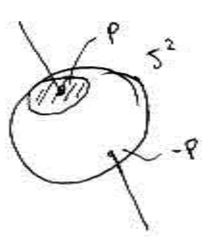
Compatibility ?

$$\tilde{\pi}_y \circ \tilde{\pi}_z^{-1} : \underset{\mathbb{R}^2}{\underset{\cap}{\tilde{\pi}_z(\tilde{U}^+ \cap \tilde{V}^+)}} \longrightarrow \underset{\mathbb{R}^2}{\underset{\cap}{\tilde{\pi}_y(\tilde{U}^+ \cap \tilde{V}^+)}}$$

$$(\tilde{\pi}_y \circ \tilde{\pi}_z^{-1})(x,y) = \tilde{\pi}_y(\tilde{\pi}_z^{-1}(x,y)) = \tilde{\pi}_y(\ell \ni \ell \cap U^+ = \{(x,y)\}) = (x,z)$$

$$= (x, \sqrt{1-x^2-y^2}) \quad \text{which is smooth for our domains}$$

$$\{(\tilde{U}^+, \tilde{\pi}_z), (\tilde{V}^+, \tilde{\pi}_y), (\tilde{W}^+, \tilde{\pi}_x)\} = \mathcal{A} \quad \text{an atlas}$$



$$U \cap (-U) = \phi.$$

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$$\chi_1(U_1 \cap U_2) = \chi_2(U_1 \cap U_2) = \left\{ (a,b) \in S \;\middle|\; \begin{matrix} -5 < a < -4 \\ \text{or} \\ 4 < a < 5 \end{matrix} \right\}$$

$$(\chi_1 \circ \chi_2^{-1})(\Delta) = \chi_1([\Delta, 2]) = \chi_1([F(\Delta), 1]) = F(\Delta)$$

$$(\chi_2 \circ \chi_1^{-1})(\Delta) = \chi_2([\Delta, 1]) = \chi_2([F^{-1}(\Delta), 2]) = F^{-1}(\Delta)$$

F glues a cylinder

G glues a mobius band

Spherical Coordinates

$$S(\rho, \varphi, \theta) = (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho\cos\varphi)$$

$$0 < \theta < 2\pi$$
$$0 < \varphi < \pi$$
$$\rho > 0$$

However if we allow a range for φ that includes zero then

$$S(\rho, 0, \theta) = (0, 0, \rho) \leftarrow \text{definitely not 1-1.}$$

$\therefore$ we can't cover the z-axis using straight spherical coordinates. In order to cover $\mathbb{R}^3/\{0\}$ we need to modify the reference axis of the angles

$$\tilde{S}(\rho, \varphi, \theta) = (\rho \sin\varphi\cos\theta, \rho\cos\varphi, \rho\sin\theta\sin\varphi)$$

Homework

1.2.4
1.2.9
1.3.3
1.3.4

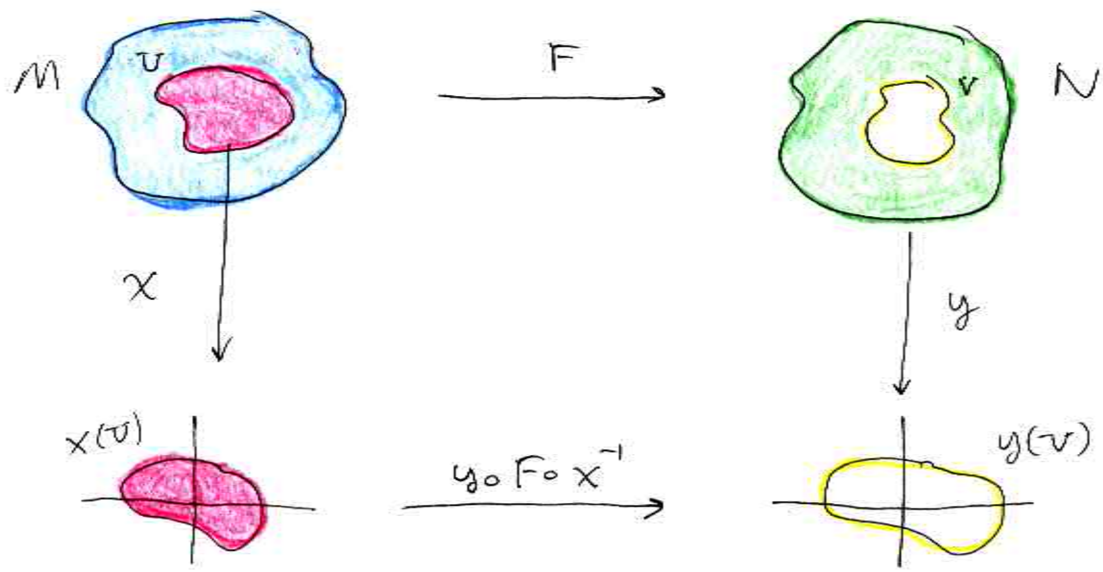
Due Friday

Theorem

If M and N are manifolds and $F: M \to N$ where $\mathcal{a}_M$ and $\mathcal{a}_N$ are atlases of M and N respectively. Then F is smooth iff $\forall\ (\mathcal{U}, x) \in \mathcal{a}_M$, $(\mathcal{V}, y) \in \mathcal{a}_N$, $y \circ F \circ x^{-1}$ is smooth.

Proof

First assume F is smooth. If $(\mathcal{U}, x) \in \mathcal{a}_M$ and $(\mathcal{V}, y) \in \mathcal{a}_N$ then we know $(\mathcal{U}, x) \in \mathcal{a}_M^*$ and $(\mathcal{V}, y) \in \mathcal{a}_N^*$ thus $y \circ F \circ x^{-1}$ is smooth



Conversely assume $y \circ F \circ x^{-1}$ is smooth $\forall\ (\mathcal{U}, x) \in \mathcal{a}_M$ and $(\mathcal{V}, y) \in \mathcal{a}_N$. Let $(\tilde{\mathcal{U}}, \tilde{x}) \in \mathcal{a}_M^*$ and $(\tilde{\mathcal{V}}, \tilde{y}) \in \mathcal{a}_N^*$. We show then that $\tilde{y} \circ F \circ \tilde{x}^{-1}$ is smooth. Assume then that $u_0 \in \text{domain}(\tilde{y} \circ F \circ \tilde{x}^{-1})$.