Manifolds, Tensor Analysis, and Applications
Abraham, Marsden, Ratiu

Let $P = \tilde{x}^{-1}(u_0)$ and $q = F(P)$. Choose $(U, x) \in \mathcal{A}_M$ and $(V, y) \in \mathcal{A}_N$ such that $p \in U$ and $q \in V$.

$$\tilde{y} \circ F \circ \tilde{x}^{-1} = (\tilde{y} \circ y^{-1}) \circ (y \circ F \circ x^{-1}) \circ (x \circ \tilde{x}^{-1})$$

Is smooth at u_0. But since u_0 was arbitrairy point in $\mathrm{dom}(\tilde{y} \circ F \circ \tilde{x}^{-1})$ we have that $\tilde{y} \circ F \circ \tilde{x}^{-1}$ is smooth since smoothness is local property. $\boxed{\text{QED}}$

THEOREM

If M, N are manifolds then $F : M \to N$ is smooth iff $\forall$ smooth real-valued function y defined on an open subset V of N then $y \circ F$ is smooth.

Manifold on an open Subset of Manifold N

$\mathcal{A}_N$ an atlas for N

$$(V \cap U, \; x|(V \cap U))$$

$$x|(V \cap U) : V \cap U \to x(V \cap U)$$

$$\mathcal{A}_V = \left\{ (U \cap V, \; x|(U \cap V)) \;\middle|\; \begin{array}{l} (U, x) \in \mathcal{A}_N \\ (U \cap V) \neq \emptyset \end{array} \right\}$$

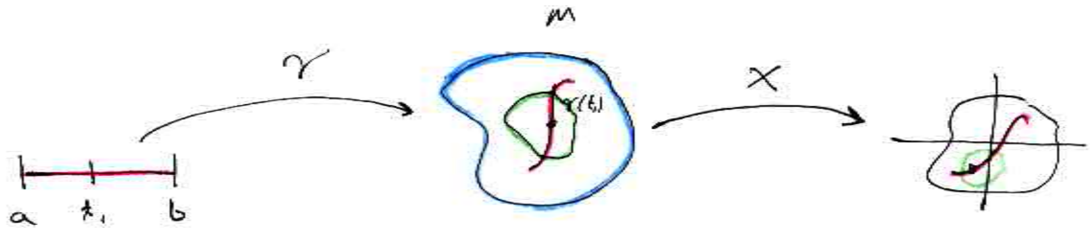
Theorem's Pf/ If $F : M \to N$ is smooth and $y : V \to \mathbb{R}$ is smooth then so is $y \circ F$ since for the charts $(U, x) \in \mathcal{A}_M$ and $(W, z) \in \mathcal{A}_N$

$$i_{\mathbb{R}} \circ (y \circ F) \circ x^{-1} = (y \circ z^{-1}) \circ (z \circ F \circ x^{-1})$$

Conversely suppose that $y \circ F$ is smooth $\forall (V, y)$. We prove F is smooth. Choose $p \in M$. Let $(U, x) \in \mathcal{A}_M$ such that $p \in U$ and $(W, z) \in \mathcal{A}_N$ such that $F(p) \in W$. We check to see if $z \circ F \circ x^{-1}$ is smooth. Let $z(w) = (z^1(w), z^2(w), \ldots, z^n(w))$ $\forall w \in \mathrm{dom}(z)$. Then $z^i : W \to \mathbb{R}$ $\forall i$ and $\therefore$ $z^i \circ F$ is smooth $\forall i$. $\therefore$ $z^i \circ F \circ x^{-1}$ is smooth $\forall (U, x)$ and $z \circ F \circ x^{-1} = (z^1 \circ F \circ x^{-1}, \ldots, z^n \circ F \circ x^{-1})$ is smooth

$\boxed{\text{QED}}$

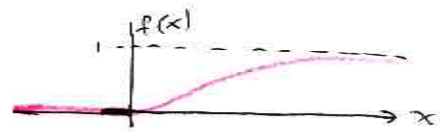
To say that $\gamma : [a,b] \longrightarrow M$ is a curve in a manifold M means that γ is continuous.



To say that $\gamma : [a,b] \longrightarrow M$ is smooth means that $\exists\ \epsilon > 0$ such that for some smooth $\tilde{\gamma} : (a-\varepsilon, b+\varepsilon) \longrightarrow M$, $\gamma = \tilde{\gamma} \big/ [a,b]$ this is what we do at the boundary points.

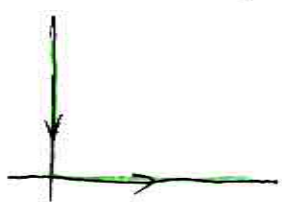
<u>Recall the interesting Hwk Problem 1.1.1</u>

$$f(x) = \begin{Bmatrix} e^{-1/x} & x > 0 \\ 0 & x \le 0 \end{Bmatrix} \qquad \lim_{x \to \infty} f(x) = 1$$



Consider then $\gamma : \mathbb{R} \to \mathbb{R}^2$

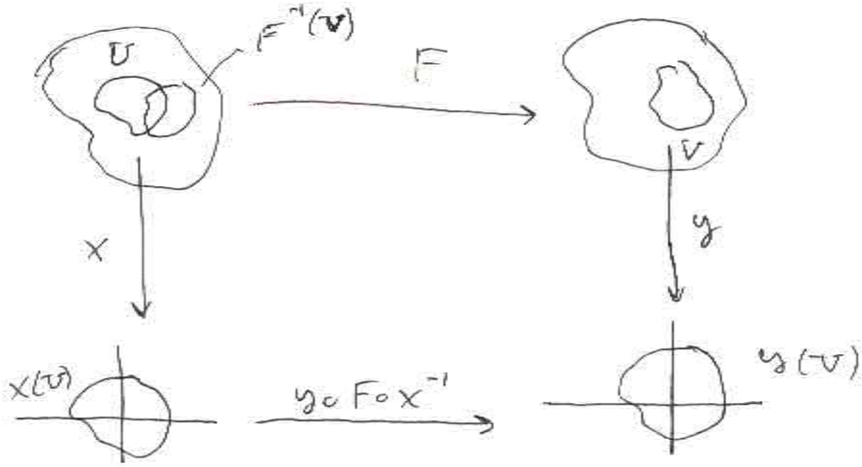
$$\gamma(x) = (f(x), f(-x))$$



Assume M and N are manifolds and $F: M \to N$. We say F is smooth iff $\forall$ pair of charts (U, x), (V, y) in M and N respectively, ~~then~~ has F_{xy} smooth.

$$y \circ F \circ x^{-1}$$

That is the local coordinate representative is smooth



$$F_{xy} : x(U \cap F^{-1}(V)) \longrightarrow y(V)$$

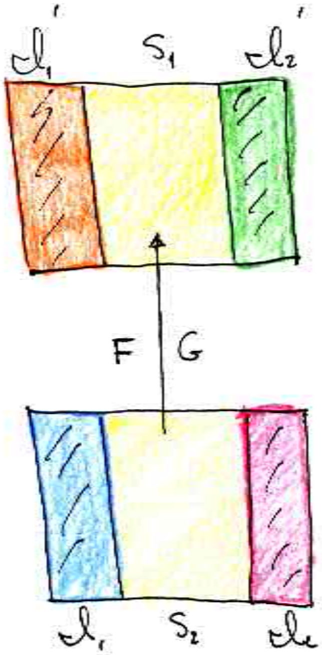
<u>Concerning the omission of the maximal atlas</u>

$a_M \subseteq a_M^*$ $\quad$ F smooth if it checks for a_M and a_N

$a_N \subseteq a_N^*$ $\quad$ type charts.

Correction of Gluing Discussion

2/1/2001



$$F(a,b)_2 = \begin{cases} (a+9,\, b)_1 & (a,b) \in \ell_1 \\ (a-9,\, b)_1 & (a,b) \in \ell_2 \end{cases}$$

$$G(a,b)_2 = \begin{cases} (a+9,\, b)_1 & (a,b) \in \ell_1 \\ (a-9,\, -b)_1 & (a,b) \in \ell_2 \end{cases}$$

$$S_1 = \{ (s,1) \mid s \in S \}$$

$$S_2 = \{ (s,2) \mid s \in S \}$$

Equivalences

$$(s,1) \sim (t,1) \iff s = t$$

$$(s,2) \sim (t,2) \iff s = t$$

$$(s,2) \sim (t,1) \iff t = F(s) \quad \text{or} \quad t = G(s)$$

$$P = S_1 \cup S_2$$

$$U_1 = \{ [s,1] \mid s \in S \} \subseteq P/\sim$$

$$U_2 = \{ [s,2] \mid s \in S \} \subseteq P/\sim$$

$$x_1 : U_1 \longrightarrow S \qquad x_1([s,1]) = s$$

$$x_2 : U_2 \longrightarrow S \qquad x_2([s,2]) = s$$

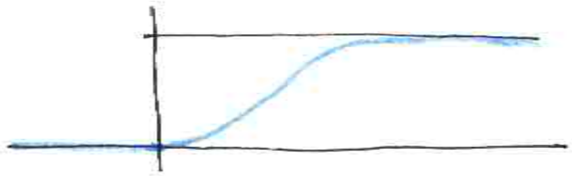
$$q \in U_1 \cap U_2 \iff q \in [s,1] = [t,2] \;,\; \exists\, s,t \in S$$

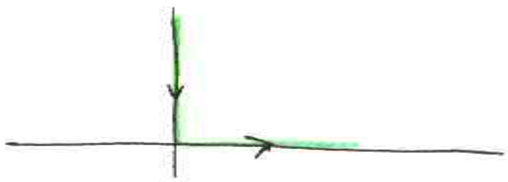
$$\iff q = [F(t),1] = [t,2] \;,\; t \in \ell$$

2/5/2001

Interesting Tricks; Picture not in book

$$f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \le 0 \end{cases}$$

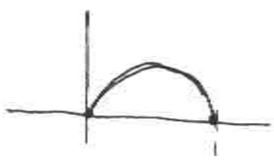


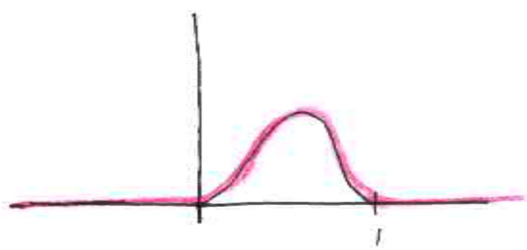


$$h(x) = \begin{cases} ce^{-\frac{1}{x(x-1)}} \\ 0 \quad x \ge 1 \\ 0 \quad x \le 0 \end{cases}$$

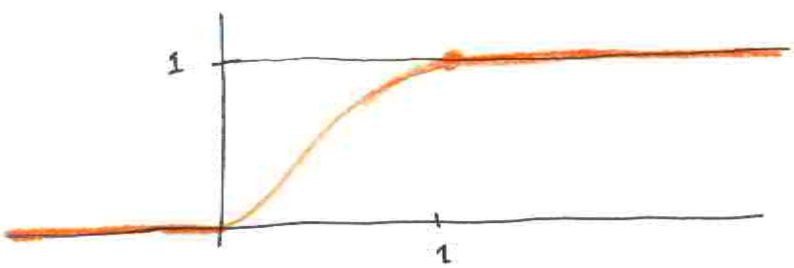
$1-x$ because $x \in (0,1)$

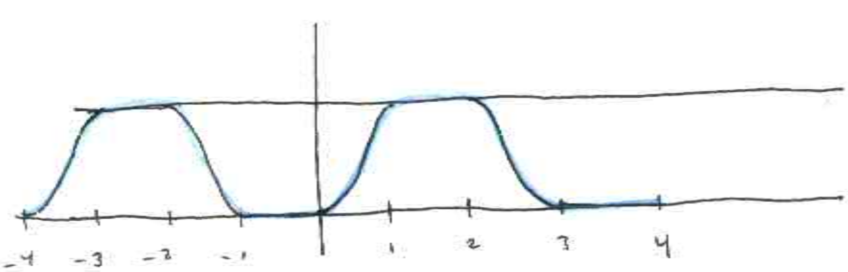
$y = x(x-1)$





$$g(x) = \int_0^x h(t)\,dt$$



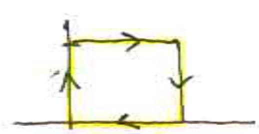


Due Tuesday
1.2.4
1.2.9
1.3.3
1.3.4

Consider 1.3.9, 1.3.10

Friday Week → 1.6.1, 1.6.2

$$f(x) = g(x) - g(x-2)$$
$$0 \le x \le 4$$
$$\gamma(x) = (f(x), f(x+1))$$



THEOREM (OR DEF^{N} DEPENDING ON YOUR BACKGROUND) 2/6/2001

a topological space X is connected $\Longleftrightarrow$ X has no nonempty subset which is both open and closed except X itself

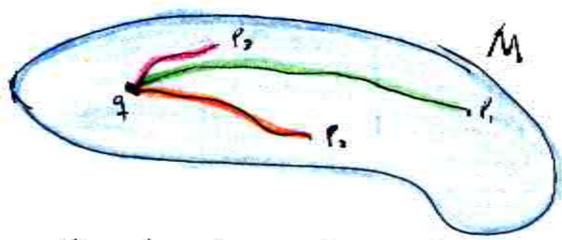
THEOREM

$$\boxed{\text{If } M \text{ is a manifold then } M \text{ is connected } \Longleftrightarrow \text{ it is pathwise connected.}}$$

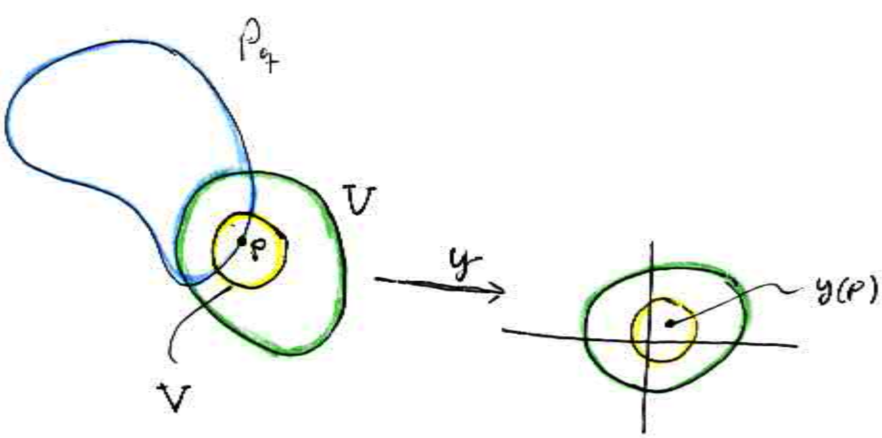
Proof Assume M is connected we show M is pathwise connected.
Let $q \in M$, Let then construct P_q the set of points pathconnected to q.

$$P_q = \{ p \in M \mid \exists \text{ a curve } \gamma : [0,1] \xrightarrow{\text{continuous}} M \text{ such that } \gamma(0) = q, \gamma(1) = p \}$$

We need to prove $P_q = M$. $q \in P_q$. First show P_q is open
Let $p \in P_q$ we show $\exists$ $V \subseteq M$ which is open such that $p \in V \subseteq P_q$



We show $\exists$ a chart (V, x) of M such that $p \in V \subseteq P_q$
Let (U, y) be a chart at p. Let B be a ball about $y(p)$
$y(p) \in \underset{\text{open}}{y(U)} \subseteq \mathbb{R}^m$ such that $B \subseteq y(U)$. Let $V = y^{-1}(B)$
and $x = y|_V$. Then (V, x) is a chart at p such that
$x(V) = B$



Let $r \in V$, $\exists\, \gamma : [0,1] \to M$ such that $\gamma(0) = q$, $\gamma(1) = p$

Since $p, r \in V$ and $x(p), x(r) \in B$. So the line from $x(p)$ to $x(r)$ is continous and lies in B. (B convex)

$$\mu(t) = t\,x(p) + (1-t)\,x(r) \qquad t \in [0,1]$$

Then $x^{-1} \circ \mu$ is a curve from p to r in V.

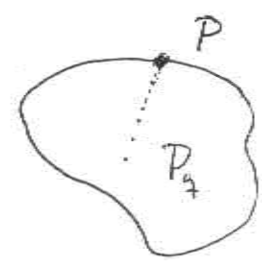
$\exists$ a continous map $\nu : [0,1] \to M$ such that

$$\nu(t) = \begin{cases} \gamma(t) & 0 \le t \le \tfrac{1}{2} \\ (x^{-1} \circ \mu)(t) & \tfrac{1}{2} < t \le 1 \end{cases}$$

So $\exists$ a curve from $q \to r$ in M $\therefore r \in P_q$

$\therefore r \in V \Rightarrow r \in P_q$ $\therefore V \subseteq P_q$ $\therefore P_q$ is open.

Now we show P_q is closed. Let $p \in M$ s.t. $\exists$ a sequence $p_n \in P_q$ such that $p_n \to p$. Thus we show every limit point of P_q is in P_q



Choose a chart (V, x) at p & $x(V)$ is a ball

Since V is open and $p \in V$ $\exists\, N \in \mathbb{N}$ s.t. $p_N \in V$

Let γ be a curve from q to p_N in M. Choose μ from $x(p_N)$ to $x(p)$ in the ball again define ν as before. Then ν is curve from q to p.

$\therefore p \in P_q$ $\therefore$ every limit point of P_q is in P_q

$\therefore P_q$ is closed.

P_q is both open and closed $\therefore P_q = M$.

$\boxed{\Leftarrow \text{ Left to reader}}$

Roger Penrose
Differential Topology of Relativity Theory

THEOREM

$\boxed{\begin{array}{l}\text{If } \exists \text{ a continous curve from } P \text{ to } q \text{ in } M, \text{ a manifold} \\ \text{then } \exists \text{ a smooth curve from } P \text{ to } q \text{ in } M.\end{array}}$

Lemma: If $P_1, P_2, \ldots, P_n$ are points in $\mathbb{R}^n$ then $\exists$ a smooth curve $\gamma : [\alpha, \beta] \longrightarrow \mathbb{R}^n$ and a sequence of numbers $\alpha = t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = \beta$ such that

$$\gamma([t_0, t_1]) = [P_1, P_2]$$

← denoting set of points lying on straight line from P_1 and P_2

$$\gamma([t_1, t_2]) = [P_2, P_3]$$

$$\vdots$$

$$\gamma([t_{n-2}, t_{n-1}]) = [P_{n-1}, P_n]$$

$$[P, q] \equiv \{(1-t)P + tq \mid 0 \leq t \leq 1\}$$

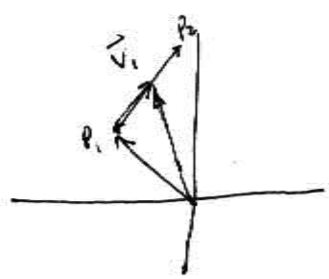
<u>Proof</u> Define $\gamma : [0, n] \longrightarrow \mathbb{R}^n$ where $\vec{V}_i \equiv \overrightarrow{P_i P_{i+1}} = P_{i+1} - P_i$

$$\gamma(t) = P_1 + g(t)\vec{V}_1 + g(t-1)\vec{V}_2 + \ldots + g(t-n+1)\vec{V}_{n-1}$$

$$g(t-k) > 0 \iff t-k > 0 \iff t > k$$

$$g(t-k) = 1 \iff t-k \geq 1 \iff t \geq k+1$$

$$\gamma(0) = P_1$$
$$\gamma(1) = P_1 + g(1)(P_2 - P_1) = P_2$$
$$\gamma(2) = P_1 + P_2 - P_1 + P_3 - P_2 = P_3$$



2/8/2001

THEOREM

> If $\exists$ a continous curve from p to q in a manifold M then $\exists$ a smooth curve from p to q in M

Pf/ Assume $\mu : [0,1] \longrightarrow M$ is continuous such that $\mu(0) = p$ and $\mu(1) = q$
For each $t \in [0,1]$ choose a chart domain U_t such that the chart x_t sends U_t onto $\mathbb{R}^m$ and such that $\mu(t) \in U_t$

DIVERSION

(V, y) then let $r \in V$ then $y(V) \underset{open}{\subseteq} \mathbb{R}^n$ $\therefore$ $\exists$ Ball B with $y(r) \in B \subseteq y(V)$

Now let $W = y^{-1}(B) \subseteq V \Rightarrow (W, y|W)$ where $y(W) = B$

$\exists$ diffeomorphism from B to $\mathbb{R}^n$ call it φ. Hint : $\varphi \propto \frac{1}{R - \|u\|^2}$

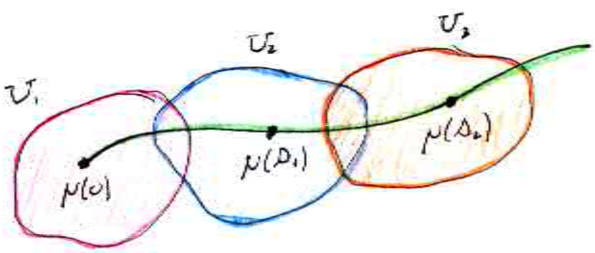
So $\varphi \circ (y|W) : W \longrightarrow \mathbb{R}^m$

$\mu[0,1]$ is compact and $\{ U_t \mid 0 \le t \le 1 \}$ covers $\mu[0,1]$. So there exists $t_1, t_2, \dots, t_n$ such that $\bigcup_{i=1}^{n} U_{t_i} \supseteq \mu[0,1]$.

Let U_1 be one of the sets $U_{t_1}, U_{t_2} \dots U_{t_n}$ which contain $\mu(0)$

Let $s_1 = \text{lub}\{ t \in [0,1] / \mu[0,t] \subseteq U_1 \}$. Now if $\mu(s_1) \in U_1$ then by continuity $\exists$ $\delta > 0$ such that $\mu((s_1 - \delta, s_1 + \delta) \cap [0,1]) \subseteq U_1$

This implies that $s_1 = 1$ and that $\mu([0, s_1]) \subseteq U_1$ $\therefore$ $\mu[0,1] \subseteq U_1$



So we see that $\mu(s) \in U_1 \Rightarrow \mu[0,1] \subseteq U_1$

Now if $\mu(s_1) \notin U_1$ then choose one of the sets $U_{t_1}, U_{t_2}, \dots, U_{t_n}$ say U_2 such that $\mu(s) \in U_2$. Then U_2 is not the same set as U_1.
Let $s_2 = \text{lub}\{ t \mid \mu([0,t]) \subseteq U_1 \cup U_2 \}$ where $t \in [0,1]$. If $\mu(s_2) \in U_1 \cup U_2$ then by continuity argument as before $s_2 = 1$ and $\mu[0,1] \subseteq U_1 \cup U_2$. If $\mu(s_2) \notin U_1 \cup U_2$ then choose one of $U_{t_1}, U_{t_2}, \dots U_{t_n}$ say U_3 such that $\mu(s_2) \in U_3$
U_3 is distinct from U_1 and U_2. Eventually we shall either find that $\exists$ s_i and U_i constructed in the obvious way.

$\mu(s_1) \notin U_1 \qquad \mu(s_1) \in U_1 \cup U_2$

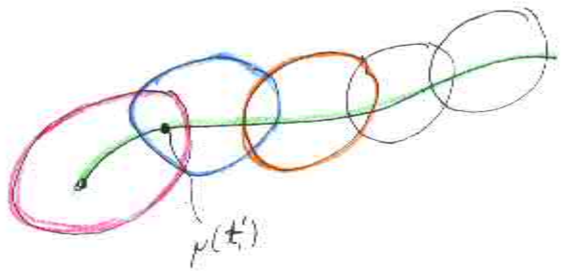
$\mu(s_2) \notin U_1 \cup U_2 \qquad \mu(s_2) \in U_1 \cup U_2 \cup U_3$

$\vdots$

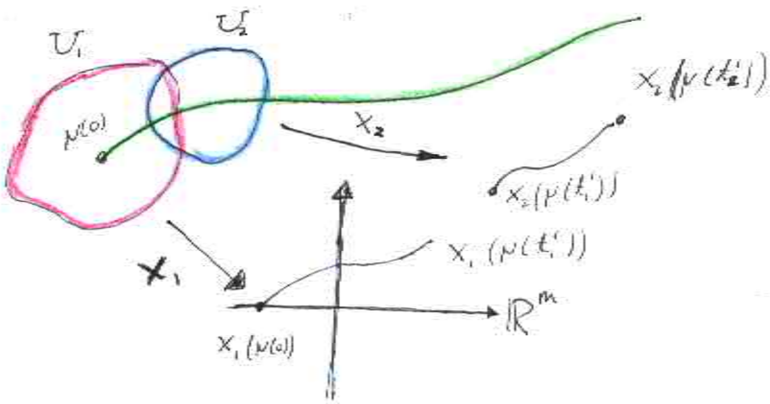
$\mu(s_N) \in U_N \qquad s_N = 1$

Continuing, $\exists\, t_1' \in [0,1]$ such that $0 \le t_1' \le s_1$ such that $\mu(t_1') \in U_1 \cap U_2$

We are trying to get a piece in both chart domains, Now $\mu(s_2) \in U_3 \;\therefore\; \exists\, t_2' \in [0,1]$



such that $\mu(t_2') \in U_3 \cap (U_1 \cap U_2)$. Then $\mu(t_i') \in U_{i+1} \cap (U_1 \cup \cdots \cup U_{i-1})$. So denote the chart defined on U_i by $x_i : U_i \to \mathbb{R}^m$.



$\tilde{x}_1 = x_1$

$\tilde{x}_2(u) = x_2(u) + x_1(\mu(t_1')) - x_2(\mu(t_1'))$

$\tilde{x}_2(\mu(t_1)) = \tilde{x}_1(\mu(t_1'))$

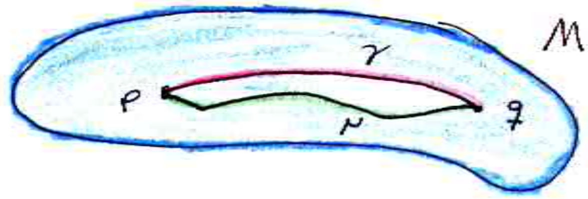
2/13/2001

THEOREM

Let $\mu : [0,1] \longrightarrow M$ be a continuous map into a manifold M.

THEN $\exists$ a smooth map $\gamma : [0,1] \rightarrow M$ such that

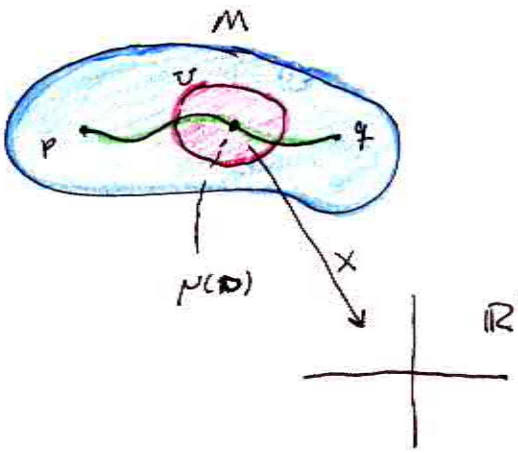
$\gamma(0) = \mu(0)$ and $\gamma(1) = \mu(1)$

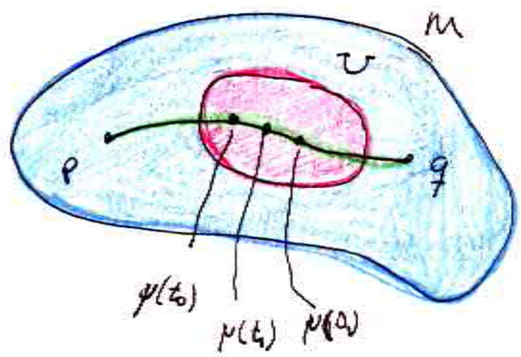


Proof

Let $V = \{ t \in [0,1] \mid \exists \text{ smooth } \gamma : [0,1] \rightarrow M \}$ such that $\gamma(0) = \mu(0)$, $\gamma(t) = \mu(t)$

Let $s = \text{lub } V$. I want to show $s \in V$ and $s = 1$. Assume that $s < 1$ or $s = 1 \notin V$. Try for a contradiction. Choose chart (U, x) at $\mu(s)$ such that $x(U) = \mathbb{R}^m$





$\exists\, t_0 < s$ such that $\mu[t_0, s] \subseteq U$

$\exists\, t_1 \in V$ such that $t_0 < t_1 < s$

$\exists$ smooth curve $\gamma : [0, t_1] \longrightarrow M$ such that $\gamma(0) = \mu(0)$, $\gamma(t_1) = \mu(t_1)$

- Def^n is curve smooth on closed interval $\Leftrightarrow$ it is continuous on open interval containing interval, for $[a,b] \in \tau \Leftrightarrow [a,b] \subset (a-\delta, b+\delta) \in \tau$.

Thus $\exists\, \delta > 0 \ni \gamma$ can be smoothly extended to $(-\delta, t_1 + \delta)$ and δ can be chosen $\ni t_1 + \delta < s$ and $\ni$

$\gamma(t_1 - \delta, t_1 + \delta) \subseteq U$