Rod Length L

$\boxed{1.2.6}$ $(u_1, u_2, u_3, u_4, u_5, u_6) \in \mathbb{R}^6$

$L(a_1, a_2, a_3)$

Show manifold same as $\mathbb{R}^3 \times S^2$

So he wants a mapping $\varphi : \mathbb{R}^3 \times S^2 \longrightarrow \mathbb{R}^6$

Show φ a 1-1 mapping. We set up a 1-1 correspondence between configuration space and manifold

I. TEST - 2 weeks from 2/15/2001

II. Homework Next Thursday

1.7.1
1.7.2
1.8.3
1.8.4

$\boxed{1.2.4}$

EASY : $\mathbb{R}^3 \times S^2 \times (L_1, L_2)$

Projective Plane Problems

$$\mathbb{R}P^2 = \{\{-P, P\} \mid P \in S^2\}$$

$$P \in S^2 \iff x(P)^2 + y(P)^2 + z(P)^2 = 1$$

$$\mathbb{R}P^2 = \left\{\{(x,y,z), (-x,-y,-z)\} \,\middle|\, x^2+y^2-z^2 = 1\right\}$$

$$\mu_1(\{+P, -P\}) = \left(\frac{y}{z}, \frac{z}{x}\right) = \mu_1\left(\{(x,y,z), (-x,-y,-z)\}\right)$$

$x^2+y^2+z^2 = 1$

$F : S^2 \longrightarrow \mathbb{R}P^2$ $\quad \chi$ a chart on S^2

$$\begin{tikzcd} S^2 \arrow[r, "F"] \arrow[d, "\chi"'] & \mathbb{R}P^2 \arrow[d, "\mu_1"] \\ {} \arrow[r] & {} \end{tikzcd}$$

$\chi^{-1} \circ F \circ \mu_1$ (smooth ?)

$$C_p^\infty(M) = \tilde{\mathcal{F}}^\infty(P) \quad : \quad \text{Book's Notation}$$

$$X_p : C_p^\infty M \longrightarrow \mathbb{R}$$

$$X_p(c_1 f_1 + c_2 f_2) = c_1 X_p(f_1) + c_2 X_p(f_2)$$

$$X_p(fg) = f(P)\, X_p(g) + g(P)\, X_p(f)$$

<u>Notation</u>

$$\left.\frac{\partial}{\partial x^i}\right|_p = \frac{\partial}{\partial x^i(P)} = \partial_i^x$$

$$\frac{\partial}{\partial u^i} = \partial_i$$

<u>Differentiation with respect to x^i from chart</u>

$$\boxed{\text{Def}^n / \quad \frac{\partial}{\partial x^i}(P)(f) \equiv \frac{\partial(f \circ x^{-1})}{\partial u^i}(x(P)) = \partial_i(f \circ x^{-1})(x(P))}$$

$$(fg) \circ x^{-1} = (f \circ x^{-1})(g \circ x^{-1})$$

$$\frac{\partial}{\partial x^i}(P)(fg) = \frac{\partial(fg \circ x^{-1})}{\partial u^i}(x(P)) = \frac{\partial}{\partial u^i}\left((f \circ x^{-1})(g \circ x^{-1})\right)(x(P))$$

$$= (f \circ x^{-1})(x(P)) \frac{\partial}{\partial u^i}(g \circ x^{-1})(x(P)) + (g \circ x^{-1})(x(P)) \frac{\partial}{\partial u^i}(f \circ x^{-1})(x(P))$$

$$= f(P) \frac{\partial}{\partial x^i}(P)\, g + g(P) \frac{\partial}{\partial x^i}(P)\, f$$

$$x(q) = (x^1(q), x^2(q), x^3(q), \ldots, x^m(q)) \;, \cancel{x^{-1}(q) = (}$$

$$x^j : U \longrightarrow \mathbb{R}$$

$$x^j \in C_p^\infty M$$

$$\frac{\partial}{\partial x^i}(p)(x^j) = \frac{\partial}{\partial u^i}(x^j \circ x^{-1})(x(p))$$

$$x^j(x^{-1}(u_1, u_2, \ldots, u_m)) = j^{\text{th}} \text{ coordinate of } x^{-1}(\vec{u}) = u^j$$

$$\therefore \frac{\partial}{\partial x^i}(p)(x^j) = \frac{\partial}{\partial u^i}(x^j \circ x^{-1})(x(p))$$

$$= \frac{\partial}{\partial u^i}(u^j)(x(p))$$

$$= \delta_i^j$$

$$\therefore \boxed{\frac{\partial x^j}{\partial x^i} = \delta_i^j}$$

OPERATIONS ON X_p's : $C_p^\infty M \to \mathbb{R}$

$$X_p, Y_p : C_p^\infty M \longrightarrow \mathbb{R}$$

$$(X_p + Y_p)(f) = X_p(f) + Y_p(f)$$

$$(c X_p)(f) = c X_p(f)$$

$$X_p = \sum_{i=1}^m a^i \frac{\partial}{\partial x^i}(p) \qquad Y_p = \sum_{j=1}^m b^j \frac{\partial}{\partial y^j}(p)$$

$$\mathbb{X}(P) = \sum_{i=1}^{m} a^i(P) \frac{\partial}{\partial x^i}(P)$$

this is a mapping from $U \subseteq M$
$U \to \bigcup_{p \in U} T_p M$

We have a finite basis for a ∞ dimensional function space ?

$a^i \in C^\infty(U) \leftarrow$ a ring but not a field. This is an ∞ dim vector space but a finitely generated module.

$$\mathbb{X} = \sum a_i \frac{\partial}{\partial x^i}$$

//

Text: t $\quad tf \to t(f)$

Fulp : $\mathbb{X}_p$ $\quad \mathbb{X}_p(f)$

Proposition :

$\boxed{\text{If } f(x) = C \text{ is constant on an open set } U \subseteq M \text{ and } \mathbb{X}_p \in T_pM \text{ then } \mathbb{X}_p(f) = 0}$

$1_m(P) = 1 \quad \forall P \in M$

Pf/ $\mathbb{X}_p(f) = \mathbb{X}_p(1_m \cdot f) = 1_m(P)\mathbb{X}_p(f) + f(P)\,\mathbb{X}_p(1_m)$

$= \mathbb{X}_p(f) + c\,\mathbb{X}_p(1_m)$

$\therefore \quad c\,\mathbb{X}_p(1_m) = 0 \quad \therefore \quad \mathbb{X}_p(c) = \mathbb{X}_p(f) = 0.$

$\boxed{\text{Proposition: If } f, g \in C_p^\infty(M) \text{ and } \exists \text{ open set } U \text{ such that } p \in U \text{ and } f(x) = g(x) \ \forall x \in U \text{ then } \mathbb{X}_p(f) = \mathbb{X}_p(g)}$

Pf/ Note that $1_U \in C_p^\infty(M)$ and $1_U f = 1_U g$

$\mathbb{X}_p(1_U f) = 1_U(P)\,\mathbb{X}_p(f) + f(P)\mathbb{X}_p(1_U)$

$= \mathbb{X}_p(f) + f(P)\mathbb{X}_p(1_U)$

$\mathbb{X}_p(1_U g) = 1_U\,\mathbb{X}_p(g) + g(P)\mathbb{X}_p(1_U) = \mathbb{X}_p(g) + g(P)\mathbb{X}_p(1_U)$

$\therefore \quad \mathbb{X}_p(f) = \mathbb{X}_p(g)$

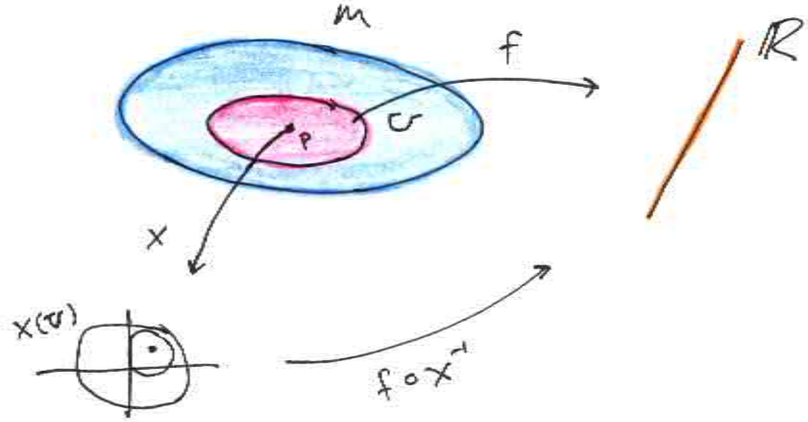
2/15/2006

Lemma: If $f \in C^\infty_p(M)$ then $\forall$ charts (U, x) at p, $U \subseteq \operatorname{dom} f$ $\exists$ a ball $B(x(p)) \subseteq x(U)$ and a smooth function $h : B(x(p)) \longrightarrow \mathbb{R}^m$ such that ~~$f(q) = f(p) + \sum_{i=1}^m h_i(x$~~

$$f(q) = f(p) + \sum_{i=1}^{m} h_i(x(q))\left[x^i(q) - x^i(p)\right]$$

for all $q \in x^{-1}(B(x(p)))$

Pf/



Let $g = f \circ x^{-1}$ and let $u_0 = x(p)$

Choose any ball $B(u_0) \subseteq x(U)$. For $u \in B(u_0)$

$$F(t) = g\big((1-t)u_0 + tu\big) \equiv g(\ell(t))$$

$$F'(t) = \sum_i \frac{\partial g}{\partial u^i}(\ell(t))\frac{d\ell^i}{dt}(t) = \sum_{i=1}^{m} \frac{\partial g}{\partial u^i}(\ell(t))\left[u^i - u_0^i\right]$$

Where $\ell^i(t) = (1-t)u_0^i + tu^i$. Integrate to find

$$\int_0^1 F'(t)\,dt = \sum_i \int_0^1 \frac{\partial g}{\partial u^i}(\ell(t))\,dt\,(u^i - u_0^i)$$

Define $h^i(u) = \int_0^1 \frac{\partial g}{\partial u^i}\big((1-t)u_0 + tu\big)dt$

Thus $g(u) - g(u_0) = \sum_i h^i(u)(u^i - u_0^i)$

2/20/2001 ①

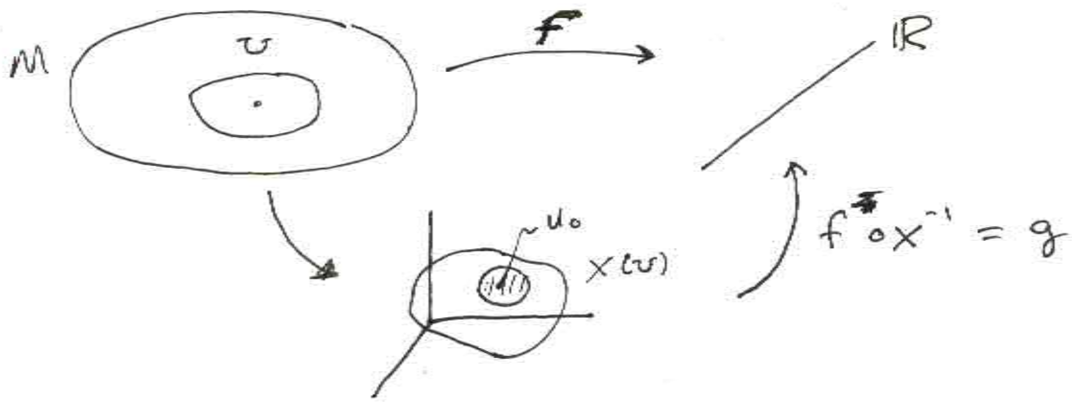
Lemma :

If $f \in C_p^\infty M$, then $\forall$ chart (U, x) at p $\exists$ a ball $B(x(p))$ in $x(U)$ and a smooth function $h : B(x(p)) \longrightarrow \mathbb{R}^m$ such that

$$\boxed{f(q) = f(p) + \sum_{i=1}^{m} h_i(x(q))\left[x^i(q) - x^i(p)\right]}$$

$\forall q \in x^{-1}(B(x(p))) \subseteq M$.

Pf//



$g = f \circ x^{-1}$ where $u_0 = x(p)$ then, $u, u_0 \in \mathbb{R}^m$,

$$F(t) = g(u_0(1-t) + tu) = g(\ell(t))$$

$$F'(t) = \sum_{i=1}^{m} \frac{\partial g}{\partial u^i}(\ell(t))\left(-u_0^i + u^i\right)$$

$$\int_0^1 F'(t)\,dt = F(1) - F(0) = g(u) - g(u_0)$$

$$g(u) - g(u_0) = \int \sum_{i=1}^{m} \frac{\partial g}{\partial u^i}(\ell(t))\left[-u_0^i + u^i\right]$$

$$= \sum_{i=1}^{m} (u^i - u_0^i) \boxed{\int \frac{\partial g}{\partial u^i}(t(u-u_0) + u_0)\,dt}$$

$$= \sum_{i=1}^{m} (u^i - u_0^i)\, h_i(u)$$

$$\therefore\ g(u) = g(u_0) + \sum_{i=1}^{m} h_i(u)\left(u^i - u_0^i\right)$$

$$\therefore\ g(x(q)) = g(x(p)) + \sum_{i=1}^{m} h_i(x(q))\left[x^i(q) - x^i(p)\right]$$

$$\Rightarrow \boxed{f(q) = f(p) + \sum_{i=1}^{m} h_i(x(q))\left[x^i(q) - x^i(p)\right]}$$

QED

Theorem

For $f \in C^\infty_p(M)$ and X_p a derivation of $C^\infty_p(M)$ it follows that

$$X_p(f) = \sum_{i=1}^{m} X_p(x^i) \frac{\partial}{\partial x^i}\Big|_p(f)$$

Now then $x : U \longrightarrow x(U) \subseteq \mathbb{R}^m$, $x(P) = (x^1(P), x^2(P), \dots, x^m(P))$ then note that $x^i : U \to \mathbb{R}$ and $x^i \in C^\infty_p M$. Further

$$X_p(f) = \sum_{i=1}^{m} X_p(x^i) \frac{\partial f}{\partial x^i}(P)$$

$$X_p = \sum_{i=1}^{m} X_p(x^i)\left(\frac{\partial}{\partial x^i}\Big|_p\right)$$

Pf/ We use the lemma, note q is a variable while P is fixed, write as a pure function eq^{n}.

$$f = f(P) + \sum_{i=1}^{m} (h_i \circ x)\left[x^i - x^i(P)\right]$$

Derivations act on $C^\infty_p M$ which is good since $f \in C^\infty_p M$,

$$X_p(f) = X_p(f(P)) + X_p\Big(\sum_{i=1}^{m} (h_i \circ x)(x^i - x^i(P))\Big)$$

Now then $X_p(\text{constant}) = 0 \therefore X_p(f(P)) = 0$. Thus using Leibniz formula

$$X_p(f) = \sum_{i=1}^{m} X_p\big((h_i \circ x)(x^i - x(P))\big)$$

$$= \sum_i \Big\{ h_i(x(P))\, X_p[x^i - \underbrace{x^i(P)}_{\text{zero}}] + \underbrace{(x^i - x^i(P))(P)}_{\text{zero.}}\, X_p(h_i \circ x)\Big\}$$

$$= \sum_{i=1}^{m} \boxed{h_i(x(P))}\, X_p(x^i)$$

So this had better be $\frac{\partial f}{\partial x^i}(P)$

Thus using what was just proven. Insert $\frac{\partial}{\partial x^i}\Big|_p \to X_p$ of 1^{st} part

$$\frac{\partial}{\partial x^i}\Big|_p (f) = \sum_{j=1}^{m} h_j(x(P)) \frac{\partial}{\partial x^i}\Big|_p (x^j)$$

$$= \sum_{j=1}^{m} h_j(x(P))\ \delta_i^j$$

$$= h_i(x(P))$$

$$\therefore\ X_p(f) = \sum_{i=1}^{m} \frac{\partial}{\partial x^i}\Big|_p (f)\ X_p(x^i)$$

$$\boxed{X_p = \sum_{i=1}^{m} X_p(x^i)\left(\frac{\partial}{\partial x^i}\Big|_p\right)}$$

$\boxed{\text{QED}}$

in other class $X_x^i = X_p(x^i)$

Changing Charts

$$X_p = \sum_{i=m}^{m} X_p(x^i) \frac{\partial}{\partial x^i}\Big|_p \quad \longleftarrow \text{ in chart } (U, x)$$

$$X_p = \sum_{j=1}^{m} X_p(y^j) \frac{\partial}{\partial y^j}\Big|_p \quad \longleftarrow \text{ in chart } (V, y)$$

How is one basis expansion related to the other? Now $\&$ $y^j \in C_p^\infty M$ and X_p acts on $C_p^\infty M$ so

$$X_p(y^j) = \sum_{i=1}^{m} X_p(x^i) \frac{\partial}{\partial x^i}\Big|_p (y^j)$$

$$= \sum_{i=1}^{m} X_p(x^i) \frac{\partial y^j}{\partial x^i}(P)$$

However we may write $\frac{\partial y^j}{\partial x^i}(P) = \frac{\partial(y^j \circ x^{-1})}{\partial u^i}$

$x(U) \xrightarrow{x^{-1}} U \xrightarrow{y^j} \mathbb{R}$ happily we recall $J_{y\circ x^{-1}}(x(P))^j_i = \frac{\partial(y^j \circ x^{-1})}{\partial u^i}$

$$X_p(y^j) = \sum_{i=1}^{m} X_p(x^i)\, J_{y\circ x^{-1}}(x(P))^j_i$$

aka $\boxed{X_y^j = X_x^i\, J_{y\circ x^{-1}}(x(P))^j_i}$ $\longleftarrow$ Σ implied over i

TANGENTS IN MANY CLOTHINGS

$$D_u f = (\nabla f \cdot u) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\cdot(u^1, u^2) = u^1\frac{\partial f}{\partial x} + u^2\frac{\partial f}{\partial y} = \left[u^1\frac{\partial}{\partial x} + u^2\frac{\partial}{\partial y}\right] f$$

$$D_u = u^1\frac{\partial}{\partial x} + u^2\frac{\partial}{\partial y}$$

typically in calculus we take u to be unit as to find correct rate of change in the direction $\hat{u}$ but in manifold theory we don't necessarily have a metric so we let u have any length,

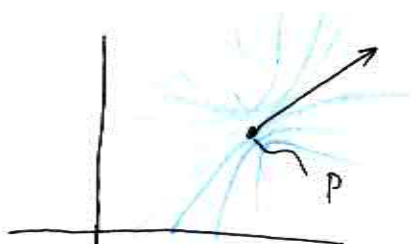
$$X_p = \sum a_x^i \frac{\partial}{\partial x^i}\bigg|_p \longleftrightarrow (a_x^1, a_x^2, \ldots, a_x^m)$$

suppose x is rectangular coordinates then we might represent

(a_x^1, a_x^2) and for polar $(a^1_{(r,\theta)}, a^2_{(r,\theta)})$

$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ basis $\qquad \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}$ basis

How do these representations relate? just like $X_y^j = \sum_i X_x^i J_{y\circ x^{-1}}(x(p))^j_i$.



$\exists$ a curve that goes through p with slope in direction of arrow. But there are many curves thus we use a equivalence class of curves.

Let us identify the tangent structures, first define $\mathcal{C}_p$

$$\mathcal{C}_p = \{\gamma \mid \exists a > 0 \text{ such that } \gamma : (-a, a) \to M \text{ is smooth and } \gamma(0) = p$$

$$\mathcal{D}_p = \{ X_p \mid X_p \text{ is a derivation of } C_p^\infty M\}$$

We define Δ to show equivalence of $\mathcal{C}_p$ and $\mathcal{D}_p$ structures,

$$\Delta : \mathcal{C}_p \longrightarrow \mathcal{D}_p \quad \text{by} \quad \Delta(\gamma) = \sum_{i=1}^{m} \frac{d(x^i \circ \gamma)}{dt}(0)\left(\frac{\partial}{\partial x^i}\bigg|_p\right)$$

$$\Delta(\gamma) = \sum_{i=1}^{m} \underbrace{(x^i \circ \gamma)'(0)}_{\mathbb{X}_x^i} \left(\frac{\partial}{\partial x^i}\Big|_p\right) = \sum_{i=1}^{m} \sum_{j=1}^{m} \underbrace{\frac{d(y^j \circ \gamma)}{dt}(0) \frac{\partial x^i}{\partial y^j}}_{\mathbb{X}_y^j} \left(\frac{\partial}{\partial x^i}\Big|_p\right)$$

Therefore continuing

$$\Delta(\gamma) = \sum_j \frac{d(y^j \circ \gamma)}{dt}(0) \sum_i \frac{\partial x^i}{\partial y^j} \left(\frac{\partial}{\partial x^i}\Big|_p\right) = \sum_j \frac{d(y^j \circ \gamma)}{dt}(0) \frac{\partial}{\partial y^j}\Big|_p$$

$$\sum_j \frac{\partial x^i}{\partial y^j} \frac{\partial}{\partial x^i}\Big|_p f = \sum_j \frac{\partial (x^i \circ y^{-1})}{\partial v^j} \frac{\partial (f \circ x^{-1})}{\partial u^i}$$

$$= \sum_j \frac{\partial (f \circ x^{-1})}{\partial u^i} \frac{\partial (x^i \circ y^{-1})}{\partial v^j}$$

$$= \sum_j \frac{\partial (f \circ y^{-1})}{\partial v^j}$$

$$= \sum_j \frac{\partial f}{\partial y_j} \qquad \therefore \sum_j \frac{\partial x^i}{\partial y^j} \frac{\partial}{\partial x^i}\Big|_p = \sum_j \frac{\partial}{\partial y^j}\Big|_p$$

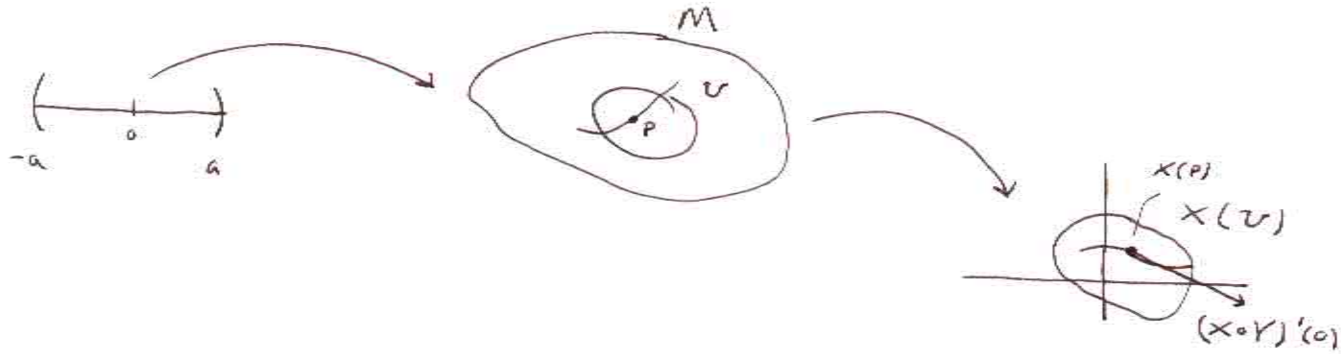
$$\mathcal{C}_p = \{\gamma \mid \exists a > 0 \text{ such that } \gamma : (-a, a) \to M \text{ smooth with } \gamma(0) = p\}$$

$$\mathcal{D}_p = \{X_p \mid X_p \text{ is a derivation of } C_p^\infty M\}$$

$$\Delta : \mathcal{C}_p \longrightarrow \mathcal{D}_p$$

$$\Delta(\gamma) = \sum_{i=1}^{m} \frac{d(x^i \circ \gamma)}{dt}(0) \left(\frac{\partial}{\partial x^i}\Big|_p\right) = \sum_{j=1}^{m} \frac{d(y^j \circ \gamma)}{dt}(0) \left(\frac{\partial}{\partial y^j}\Big|_p\right)$$

Here (U, x) and (V, y) are charts on M at p.



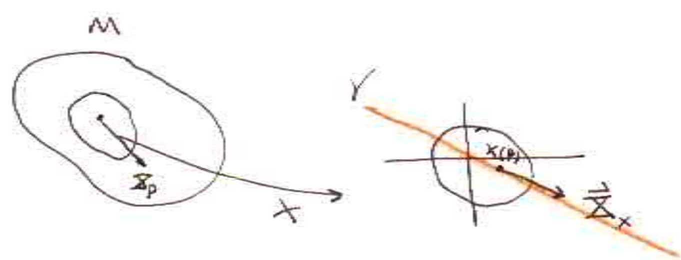
$t \longrightarrow (x^1(\gamma(t)), x^2(\gamma(t)), \ldots)$

We first show that Δ is onto.

Let $X_p \in \mathcal{D}_p$. Choose a chart (U, x) at p and write then

$$X_p = \sum_{i=1}^{m} X_x^i \left(\frac{\partial}{\partial x^i}\Big|_p\right) \quad \text{where} \quad X_x^i = X_p(x^i)$$

We take the derivation in M we may think of it in $\mathbb{R}^m$



$$X_p = \sum X_x^i \frac{\partial}{\partial x^i}\Big|_p \in T_p M \quad , \quad \vec{X}_x = (X_x^1, \ldots, X_x^m)$$

We want a curve so that $X_x^i = \frac{d(x^i \circ \gamma)}{dt}(0)$

We take the simplest curve γ as the curve which satisfys

Define γ by

$$\gamma(t) = x^{-1}\left(x(P) + t\,\overline{\Xi}_x\right) = x^{-1}\left(x(P) + t\sum_{i=1}^{m} \Xi_x^i e_i\right) \quad \text{(in component)}$$

Then we find $x \circ \gamma(t)$, $x \circ x^{-1} = \text{identity}$ thus,

$$x \circ \gamma(t) = x(P) + t\left(\Xi_x^1, \Xi_x^2, \ldots, \Xi_x^m\right)$$

$$x^i(\gamma(t)) = x^i(P) + t\,\Xi_x^i$$

$$\therefore \frac{d(x^i \circ \gamma)}{dt}(0) = \Xi_x^i$$

$$\therefore \Delta(\gamma) = \sum_{i=1}^{m} \frac{d(x^i \circ \gamma)}{dt}(0)\left(\frac{\partial}{\partial x^i}\Big|_P\right) = \sum_{i=1}^{m} \Xi_x^i \left(\frac{\partial}{\partial x^i}\Big|_P\right) = \Xi_P$$

$\therefore$ Δ is onto.

If $\gamma_1, \gamma_2 \in \mathcal{C}_P$ and $\Delta(\gamma_1) = \Delta(\gamma_2)$ then in any chart (U, x)

$$\sum_i \frac{d(x^i \circ \gamma_1)}{dt}(0)\left(\frac{\partial}{\partial x^i}\Big|_P\right) = \sum_i \frac{d(x^i \circ \gamma_2)}{dt}(0)\left(\frac{\partial}{\partial x^i}\Big|_P\right)$$

These are derivations, they act on $C_P^\infty(M)$ then $x^j \in C_P^\infty M$ so we act on it,

$$\sum_i \frac{d(x^i \circ \gamma_1)}{dt}(0)\frac{\partial x^j}{\partial x^i}(P) = \sum_i \frac{d(x^i \circ \gamma_2)}{dt}(0)\frac{\partial x^j}{\partial x^i}(P)$$

$$\therefore \frac{d(x^j \circ \gamma_1)}{dt}(0) = \frac{d(x^j \circ \gamma_2)}{dt}(0)$$

$$\frac{d}{dt}\left(x^1(\gamma_1(t)), x^2(\gamma_1(t)), \ldots, x^m(\gamma(t))\right)(0) = \frac{d}{dt}\left(x^1(\gamma_2(t)), \ldots, x^m(\gamma_2(t))\right)(0)$$

$$\therefore \frac{d(x \circ \gamma_1)}{dt}(0) = \frac{d(x \circ \gamma_2)}{dt\,dt}(0) \Rightarrow (x \circ \gamma_1)'(0) = (x \circ \gamma_1)'(0)$$

This implies $\gamma_1 \sim \gamma_2$ or that $[\gamma_1] = [\gamma_2]$ since $\gamma_1(0) = \gamma_2(0) = P$ as well.
Define $\tilde{\Delta} : \mathcal{C}_P/\sim \longrightarrow \mathcal{D}_P$ by $\tilde{\Delta}([\gamma]) = \Delta(\gamma)$ which is well defined and onto and 1-1. We have a formal equivalence of derivations and equivalence classes of curves (2 ideas of a tangent vector!)

$$T_pM = \mathcal{D}_p \cong \left(G_p/\sim \right)$$

$[\gamma] \in T_pM$

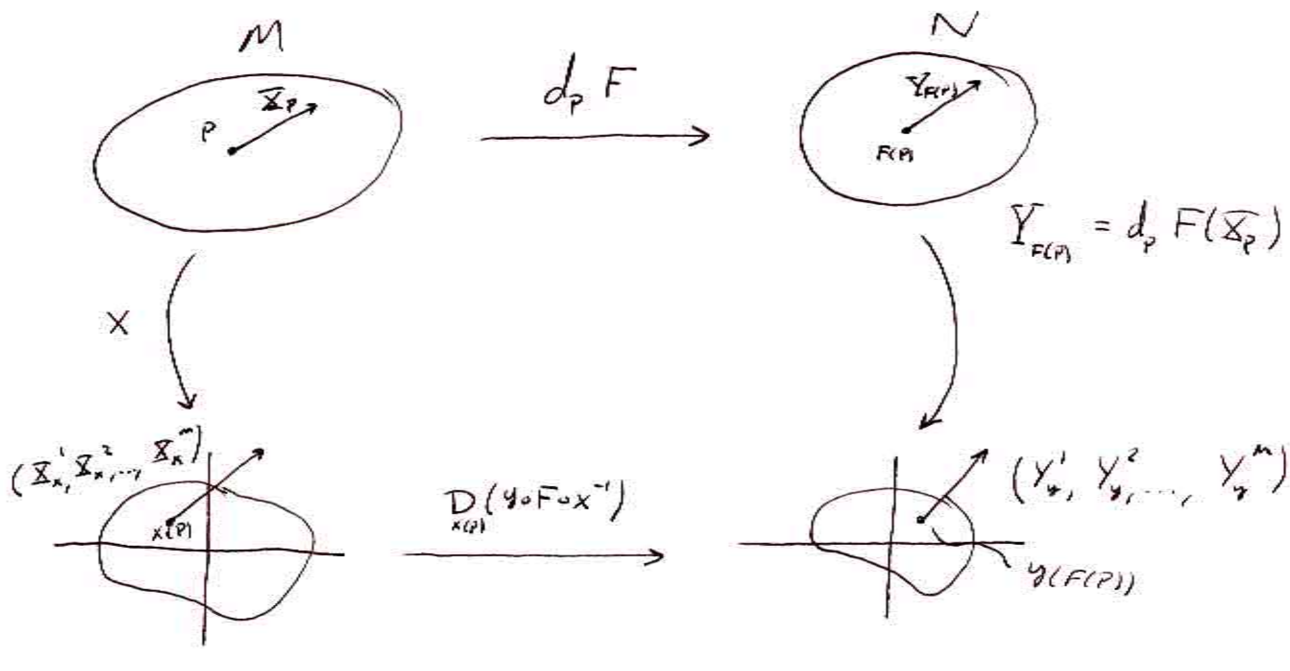
$X_p \in T_pM \Rightarrow \exists\, \gamma$ such that $\gamma'(0) = X_p$

$$\gamma'(t) = \sum \frac{d(x^i \circ \gamma)}{dt}(t) \left(\frac{\partial}{\partial x^i}\Big|_{\gamma(t)} \right)$$

Or we could conversely generate a derivation from $[\gamma]$ vise versa.

Let $F : M \longrightarrow N$, The differential of F is the map
$d_pF : T_pM \longrightarrow T_{F(p)}N$ also denoted $F_*(p)$ def^n by,

$$d_pF(X_p) = Y_p \iff D_{x(p)}(y \circ F \circ x^{-1})(X_x^1, X_x^2, \dots, X_x^m) = (Y_y^1, Y_y^2, \dots, Y_y^n)$$



If we have $g : E \to \tilde{E}$ open sets in Eucliden spaces then

$$D_u g(v^1, v^2, \dots, v^m) = (v^1, v^2, \dots, v^m)\, J_g(u)^t$$

Then the explicit calculational form of the differential

$$(Y_y^1, \dots, Y_y^m) = (\Xi_x^1, \dots, \Xi_x^m) J_{y\circ F\circ x^{-1}}(x(P))^{+}$$

or in tensorial equation,

$$Y_j^i = \sum_j \Xi_x^j \, J_{y\circ F\circ x^{-1}}(x(P))_j^i$$

Then we find,

$$\boxed{d_P F\left(\sum_i \Xi_x^i \left(\frac{\partial}{\partial x^i}\Big|_P\right)\right) = \sum_j \sum_i \Xi_x^j \left[J_{y\circ F\circ x^{-1}}(x(P))\right]_j^i \frac{\partial}{\partial y^i}\Big|_{F(P)}}$$

$$J_{y\circ F\circ x^{-1}}(x(P))_j^i \equiv \frac{\partial(y^i\circ F\circ x^{-1})}{\partial u^j}(x(P)) = \frac{\partial(y^i\circ F)}{\partial x^j}(P)$$

So in slightly condensed notation,

$$\boxed{dF\left(\Xi^i \frac{\partial}{\partial x^i}\right) = \sum_{i,j} \Xi^j \frac{\partial(y^i\circ F)}{\partial x^j}\left(\frac{\partial}{\partial y^i}\right)}$$

$$d_P F(\gamma'(0)) = \sum_{i,j} \frac{d(x^j\circ\gamma)}{dt}(0) \frac{\partial(y^i\circ F)}{\partial x^j} \frac{\partial}{\partial y^i}$$

$$= \sum_i \frac{d}{dt}(y^i\circ F\circ\gamma)(0) \frac{\partial}{\partial y^i}\Big|_{F(P)}$$

Harness the isomorphism Δ to find,

$$\therefore \boxed{d_P F(\gamma'(0)) = (F\circ\gamma)'(0)}$$

$$\boxed{d_P F([\gamma]) = [F\circ\gamma]}$$

differentiation hidden in the equivalence relation

What does $\gamma'(0)f$ mean? $f \in C_p^\infty M$.

$$\gamma'(t)(f) = \sum_i \frac{d(x^i \circ \gamma)}{dt}(t)\, \frac{\partial f}{\partial x^i}(\gamma(t)) \overset{?}{=} \frac{d}{dt}\left[f(\gamma(t))\right]$$

To show we need to insert $x x^{-1}$ as to be able to use chain rule,

$$\frac{d}{dt}\left[(f \circ \gamma)(t)\right] = \frac{d}{dt}\left[(f \circ x^{-1}) \circ (x \circ \gamma)\right](t) =$$

$$\frac{\partial (f \circ x^{-1})}{\partial u^j} \frac{d(x^j \circ \gamma)}{dt}$$

$$\therefore \quad \boxed{\gamma'(t)(f) = (f \circ \gamma)'(t)}$$

Next Topic

$$f : M \longrightarrow \mathbb{R}^n$$

$$d_p f : T_p M \longrightarrow T_{f(p)} \mathbb{R}^n = \mathbb{R}^n$$

$$v^1 \frac{\partial}{\partial x} + v^2 \frac{\partial}{\partial y} + v^3 \frac{\partial}{\partial z} \cong (v^1, v^2, v^3)$$

$$\sum_{i=1}^n v^i \frac{\partial}{\partial u^i} = (v^1, v^2, \dots, v^n) \quad \text{essentially } \frac{\partial}{\partial u^i} = e_i$$

$$d_p f\left(\sum X_x^i \left(\frac{\partial}{\partial x^i}\Big|_p\right)\right) = \sum_{i,j} X_x^j \frac{\partial u^i \circ f}{\partial x^j} \frac{\partial}{\partial u^i}$$

$$= \sum_{i,j} X_x^i \frac{\partial (u^i \circ f)}{\partial x^j} e_i$$

$$= \sum_j X_x^j \frac{\partial f^i}{\partial x^j} e_i$$

$$= \sum_j \left(X_x^j \frac{\partial f^1}{\partial x^j}, X_x^j \frac{\partial f^2}{\partial x^j}, \dots, X_x^j \frac{\partial f^m}{\partial x^j} \right)$$

So if $n = 1$ then $f : M \to \mathbb{R}$ and

$$\boxed{d_p f(X_p) = X_p(f)}$$