Problems are typically taken from either Jeffrey Lee's text *Manifolds and Differential Geometry* (MDG) or John Lee's text *Smooth Manifolds* (SM). I've also written a few problems. Most problems 5pts here.

Problem 93 If E_1, E_2, E_3 is an orthonormal frame field on $\mathbb{R}^3$ for which E_1, E_2 restrict to tangent fields to $M \subset \mathbb{R}^3$ and $E_3(p) \in T_pM^\perp$ then we say E_1, E_2, E_3 is an **adapted frame** on M.

(a.) Show $S(v) = \omega_{13}(v)E_1 + \omega_{23}(v)E_2$

(b.) Show $\omega_{13} \wedge \omega_{23} = K\theta^1 \wedge \theta^2$

(c.) Show $\omega_{13} \wedge \theta^2 + \theta^1 \wedge \omega_{23} = 2H\theta^1 \wedge \theta^2$

(d.) Show $det(S) = \omega_{13}(E_1)\omega_{23}(E_2) - \omega_{13}(E_2)\omega_{23}(E_1)$

(e.) Show $d\omega_{12} = -K\theta^1 \wedge \theta^2$,

Problem 94 A **principal frame field** adapted to M is an orthonormal frame field E_1, E_2, E_3 adapted to M for which there exist **principal curvature functions** k_1, k_2 such that $S(E_1) = k_1E_1$ and $S(E_2) = k_2E_2$. Show $E_1[k_2] = (k_1 - k_2)\omega_{12}(E_2)$ and $E_2[k_1] = (k_1 - k_2)\omega_{12}(E_1)$.

Problem 95 Consider the catenoid M given by parametric equations

$$x = b\cosh(v/b)\cos(u), \qquad y = b\cosh(v/b)\sin(u), \qquad z = v.$$

Calculate the following:

(a.) Find an adapted frame E_1, E_2, E_3 to M by normalizing $\partial_u X$ and $\partial_v X$ and setting $E_3 = E_1 \times E_2$.

(b.) Find coframe $\theta^1, \theta^2, \theta^3$ of the adapted frame, check that $\theta^3 = 0$ on M.

(c.) Find ω_{12} from solving Cartan's Structure Equations.

(d.) Calculate the Gaussian curvature from $d\omega_{12} = -K\theta^1 \wedge \theta^2$.

• **Problem 96** Let $F : M \to N$ be a smooth surface map. For each patch $X : D \to M$ consider the map $\bar{X} = F \circ X : D \to N$. Then F is a **local isometry** if and only if for each patch X we have $E = \bar{E}$ and $F = \bar{F}$ and $G = \bar{G}$ where $E = \partial_u X \bullet \partial_u X$ and $F = \partial_u X \bullet \partial_v X$ and $G = \partial_v X \bullet \partial_v X$. Use this theorem to find a local isometry of the

(a.) plane and cylinder which have patches $X(u,v) = (u,v,0)$ and $Y(u,v) = (R\cos(u/R), R\sin(u/R), v)$ respective.

(b.) the helicoid and catenoid which have patches $X(u,v) = (u\cos v, u\sin v, v)$ and $Y(u,v) = (g, h\cos v, h\sin v)$ where $g(u) = \sinh^{-1}(u)$ and $h(u) = \sqrt{1+u^2}$ (it's a little algebra, but you can take the implicit formulation provided by the theorem and make it explicit)

• **Problem 97** Consider a surface $M \subset \mathbb{R}^3$ with adapted frame E_i and coframe θ^i. In this problem we seek to argue for the intrinsic character of the Gaussian curvature:

(a.) Define $\gamma = (d\theta^1(E_1, E_2))\theta^1 + d\theta^2(E_1, E_2))\theta^2$ and show $\gamma = \omega_{12}$.

(b.) Let $F : M \to \overline{M}$ be an isometry and let $\overline{E}_j = F_*(E_j)$ for $j = 1, 2$. Show $\theta^j = F^*(\overline{\theta}^j)$ for $j = 1, 2$ and $\omega_{12} = F^*(\overline{\omega}_{12})$ where $\overline{\omega}_{12}$ is the connection form on $\overline{M}$.

(c.) Show $K = \overline{K} \circ F$

Problem 98 Let $g = \frac{4}{(1-x^2-y^2)^2}(dx^2 + dy^2)$ then you can show $E_i = \frac{1}{2}(1 - x^2 - y^2)\frac{\partial}{\partial x^i}$ is a g-orthonormal frame in the sense that $g(E_i, E_j) = \delta_{ij}$. You could also show the dual coframe of 1-forms θ^1, θ^2

$$\theta^1 = \frac{2dx}{1 - x^2 - y^2}, \qquad \& \qquad \theta^2 = \frac{2dy}{1 - x^2 - y^2}.$$

With all of this given, calculate the Gaussian curvature. The set $M = \{(x^1, x^2) \mid \|(x^1, x^2)\| < 1\}$ paired with the metric g is known as the hyperbolic disk. It is an example of how a subset of the plane can be given a non-Euclidean geometry.

- **Problem 99** Find a metric on the plane which gives a subset of the plane curvature $K = 1$

- **Problem 100** Find a metric on a subset of the sphere which makes it flat.

- **Problem 101** State the Gauss Bonnet Theorem. Also, in a surface of constant curvature what can we say about the interior angles of a triangle ?

- **Problem 102** Explain why the sphere is not isometric to the torus.

- **Problem 103** Show that the ellipsoid $\mathcal{E} = \{(x, y, z) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\}$ is mapped to the unit-sphere Σ via the map $F : \Sigma \to \mathcal{E}$ defined by $F(u, v, w) = (au, bv, cw)$. Also, explain why F is a diffeomorphism. Find the total curvature of $\mathcal{E}$ via the Gauss Bonnet Theorem as well as the observation that diffeomorphisms are also homeomorphisms and as such preserve the Euler characteristic. Remember, we already know the total curvature of Σ from our work in class.

Extra Problems you don't have to do:

(I.) MDG, Exercise 4.42 on page 168 (formula for Christoffel symbols in terms of metric)

(II.) Let $E_1 = -\sin\theta U_1 + \cos\theta U_2$ and $E_2 = U_3$ and $E_3 = \cos\theta U_1 + \sin\theta U_2$. Find the attitude matrix A for the cylindrical frame E_1, E_2, E_3 and calculate the matrix of connection forms $\omega = (dA)A^T$

(III.) In cylindrical coordinates, the frame of the above problem is simply $E_1 = \frac{1}{r}\frac{\partial}{\partial\theta}$, $E_2 = \frac{\partial}{\partial z}$ and $E_3 = \frac{\partial}{\partial r}$. This frame adapts nicely to the cylinder M given by $r = R$ where R is a fixed positive constant. Notice $E_1 = \frac{1}{R}\frac{\partial}{\partial\theta}$ and $E_2 = \frac{\partial}{\partial z}$ have coframe $\theta^1 = Rd\theta$ and $\theta^2 = dz$. We found $\omega_{13} = -d\theta = -\omega_{31}$ whereas $\omega_{12} = \omega_{21} = \omega_{23} = \omega_{32} = 0$.

(a.) Recall $S(E_1) = -\omega_{31}(E_1)E_1 - \omega_{32}(E_1)E_2$ and $S(E_2) = -\omega_{31}(E_2)E_1 - \omega_{32}(E_2)E_2$. Show E_1, E_2 is a principle frame.

(b.) Calculate H and K

(c.) Give an example of an asymptotic curve on M

(d.) Give an example of a geodesic curve on M

(IV.) Let θ^1, θ^2 be a coframe of E_1, E_2 on a surface such that $\theta^1 \wedge \theta^2 \neq 0$. Notice $f\theta^1 \wedge \theta^2 = g\theta^1 \wedge \theta^2$ implies $f = g$. Furthermore, if α is a one-form then

$$\alpha = \alpha[E_1]\theta^1 + \alpha[E_2]\theta^2$$

Use the assumptions above to find α given that $\alpha \wedge \theta^1 = A\theta^1 \wedge \theta^2$ and $\alpha \wedge \theta^2 = B\theta^1 \wedge \theta^2$. (your formula for α will involve A and B)

(V.) The helicoid has patch $X(u, v) = (u \cos v, u \sin v, bv)$ where $b > 0$. Since this is an orthogonal patch, we can use frame $E_1 = \frac{1}{\sqrt{E}} X_u$ and $E_2 = \frac{1}{\sqrt{G}} X_v$ and coframe $\theta^1 = \sqrt{E}du$ and $\theta^2 = \sqrt{G}dv$. Derive the connection form ω_{12} and the Gaussian curvature K from the equations $d\theta^1 = \omega_{12} \wedge \theta^2$, $d\theta^2 = \omega_{21} \wedge \theta^1$ and $d\omega_{12} = -K\theta^1 \wedge \theta^2$.

(VI.) The paraboloid of revolution has patch $X(u, v) = (u \cos v, u \sin v, u^2/2)$ where $b > 0$. Since this is an orthogonal patch, we can use frame $E_1 = \frac{1}{\sqrt{E}} X_u$ and $E_2 = \frac{1}{\sqrt{G}} X_v$ and coframe $\theta^1 = \sqrt{E}du$ and $\theta^2 = \sqrt{G}dv$. Derive the connection form ω_{12} and the Gaussian curvature K from the equations $d\theta^1 = \omega_{12} \wedge \theta^2$, $d\theta^2 = \omega_{21} \wedge \theta^1$ and $d\omega_{12} = -K\theta^1 \wedge \theta^2$.

(VII.) The cone has patch $X(u, v) = (u \cos v, u \sin v, au)$ where $a > 0$. Since this is an orthogonal patch, we can use frame $E_1 = \frac{1}{\sqrt{E}} X_u$ and $E_2 = \frac{1}{\sqrt{G}} X_v$ and coframe $\theta^1 = \sqrt{E}du$ and $\theta^2 = \sqrt{G}dv$. Derive the connection form ω_{12} and the Gaussian curvature K from the equations $d\theta^1 = \omega_{12} \wedge \theta^2$, $d\theta^2 = \omega_{21} \wedge \theta^1$ and $d\omega_{12} = -K\theta^1 \wedge \theta^2$.

(VIII.) Let $X(u, v) = (u, v, f(u, v))$ where f is a smooth function of u, v on $\mathbb{R}^2$. Let $M = X(\mathbb{R}^2)$. Calculate the Gaussian curvature of M. What condition on f is needed for M to be flat ?

(IX.) Let

$$X(u, v) = (u \cos(v), u \sin(v), v).$$

Calculate E, F, G and L, M, N. Use the standard formulas to calculate the Gaussian and mean curvature.

(X.) Suppose $u > 0$ and let

$$X(u, v) = (u \cos(v), u \sin(v), u^2).$$

Calculate the associated frame field E_1, E_2 as well as the dual frame θ^1, θ^2. Find the connection form ω_{12} and calculate the Gaussian curvature K. For bonus, also find ω_{12}, ω_{23} and calculate the mean curvature H.

Mission 10 Solution

$\boxed{P93}$ E_1, E_2, E_3 an orthonormal frame field on $\mathbb{R}^3$ for which $E_1, E_2 \in \mathfrak{X}(M)$ and $E_3(p) \in T_pM^\perp$ so E_1, E_2, E_3 is an adapted frame.

(a.) Show $S(v) = \omega_{13}(v)E_1 + \omega_{23}(v)E_2$

$$
\begin{aligned}
S(v) &= S\big((E_1 \cdot v)E_1 + (E_2 \cdot v)E_2\big) \\
&= -\nabla_v E_3 \qquad \text{since } E_1, E_2 \text{ is frame} \\
&= -\big((\nabla_v E_3)\cdot E_1\big)E_1 - \big((\nabla_v E_3)\cdot E_2\big)E_2 \\
&= -\omega_{31}(v)E_1 - \omega_{32}(v)E_2 \\
&= \underline{\omega_{13}(v)E_1 + \omega_{23}(v)E_2}.
\end{aligned}
$$

Here I used $\omega_{ij}(p)(v) = (\nabla_v E_i)\cdot E_j(p)$ from P88 as well as $\omega_{ij} = -\omega_{ji}$ which we showed in class. (and you can derive a/a P87 calculation)

(b.)

$$
\begin{aligned}
\omega_{13} &= \omega_{13}(E_1)\theta^1 + \omega_{13}(E_2)\theta^2 \\
\omega_{23} &= \omega_{23}(E_1)\theta^1 + \omega_{23}(E_2)\theta^2
\end{aligned}
$$

$$
\begin{aligned}
\omega_{13}\wedge\omega_{23} &= \big(\omega_{13}(E_1)\theta^1 + \omega_{13}(E_2)\theta^2\big)\wedge\big(\omega_{23}(E_1)\theta^1 + \omega_{23}(E_2)\theta^2\big) \\
&= \big(\omega_{13}(E_1)\omega_{23}(E_2) - \omega_{13}(E_2)\omega_{23}(E_1)\big)\theta^1\wedge\theta^2 \\
&= \det\begin{bmatrix} \omega_{13}(E_1) & \omega_{13}(E_2) \\ \omega_{23}(E_1) & \omega_{23}(E_2) \end{bmatrix}\theta^1\wedge\theta^2 \qquad \beta = \{E_1, E_2\} \\
&= \det(S)\,\theta^1\wedge\theta^2 \longleftarrow \underbrace{\big[[S(E_1)]_\beta \,\big|\, [S(E_2)]_\beta\big]_{\beta\beta}}_{\text{matrix of } S \text{ w.r.t. } \beta = \{E_1, E_2\} \text{ using (a.)}} \\
&= K\,\theta^1\wedge\theta^2
\end{aligned}
$$

P93 continued

(c.) $\omega_{13} \wedge \theta^2 + \theta^1 \wedge \omega_{23} =$

$= \left(\omega_{13}(E_1)\theta^1 + \omega_{13}(E_2)\theta^2\right) \wedge \theta^2 + \theta^1 \wedge \left(\omega_{23}(E_1)\theta^1 + \omega_{23}(E_2)\theta^2\right)$

$= \left(\omega_{13}(E_1) + \omega_{23}(E_2)\right)\theta^1 \wedge \theta^2$

$= \text{trace } [S]_{pp}\, \theta^1 \wedge \theta^2$

$= \underline{H\, \theta^1 \wedge \theta^2}$ (no 2 here I think)

(d.) yeah, we showed this already

(e.) Show $d\omega_{12} = -K\,\theta^1 \wedge \theta^2$

$d\omega_{12} = \omega_{13} \wedge \omega_{32}$: Gauss' Eq^{n} from Cartan's Structure Eq^{n}s.

$= -\omega_{13} \wedge \omega_{23}$: $\omega_{23} = -\omega_{32}$

$= -K\,\theta^1 \wedge \theta^2$ using part (b.)

P94 since $S_p(v) \cdot w = v \cdot S_p(w) \Rightarrow [S_p]^T = [S_p]$ the real spectral Th^{m} gives the existence of an orthonormal eigenbasis for S_p, let E_1, E_2 be the eigenbasis such that $S(E_1) = k_1 E_1$ and $S(E_2) = k_2 E_2$, oh the problem gives this much to us.

Notice $S(E_1) = \omega_{13}(E_1)E_1 + \omega_{23}(E_1)E_2 = k_1 E_1 \Rightarrow$
$S(E_2) = \omega_{13}(E_2)E_1 + \omega_{23}(E_2)E_2 = k_2 E_2$

Thus, $\omega_{13}(E_1) = k_1$ and $\omega_{23}(E_2) = k_2$

and $\omega_{13}(E_2) = 0$ and $\omega_{23}(E_1) = 0$

Therefore, $\omega_{13} = k_1\theta^1$ and $\omega_{23} = k_2\theta^2$

P99 continued

We found $\omega_{13} = k_1 \theta^1$ and $\omega_{23} = k_2 \theta^2$ where k_1 & k_2 are the principal curvature functions.

However, $\underbrace{d\omega_{13}} = \omega_{12} \wedge \omega_{23}$

$$d(k_1 \theta^1) = \omega_{12} \wedge (k_2 \theta^2)$$

$$dk_1 \wedge \theta^1 + k_1 d\theta^1 = k_2 \omega_{12} \wedge \theta^2$$

$$dk_1 \wedge \theta^1 + k_1 \omega_{12} \wedge \theta^2 = k_2 \omega_{12} \wedge \theta^2$$

$$\therefore dk_1 \wedge \theta^1 = (k_2 - k_1) \omega_{12} \wedge \theta^2$$

But, $dk_1 = dk_1(E_1)\theta^1 + dk[E_2]\theta^2$ hence,

$$dk_1(E_2)\theta^2 \wedge \theta^1 = (k_2 - k_1)\omega_{12}(E_1)\theta^1 \wedge \theta^2$$

$$\Rightarrow dk_1(E_2) = (k_2 - k_1)\omega_{12}(E_1)$$

$$\therefore \underline{E_2[k_1] = (k_2 - k_1)\omega_{12}(E_1)}.$$

Similarly, using $d\omega_{23} = \omega_{21} \wedge \omega_{13}$

$$d(k_2 \theta^2) = -\omega_{12} \wedge (k_1 \theta^1)$$

$$\vdots$$

$$\Rightarrow \underline{E_1[k_2] = (k_1 - k_2)\omega_{12}(E_2)}.$$

$\boxed{P95}$ Catenoid M given by

$$x = b\cosh(v/b)\cos u, \quad y = b\cosh(v/b)\sin u, \quad z = v$$

Calculate $X = (x, y, z)$ has $\gamma = 1/b$ (define)

(a.) $X_u = \langle -b\cosh(\gamma v)\sin u,\ b\cosh(\gamma v)\cos u,\ 0\rangle$

$$\frac{X_u}{\| X_u \|} = E_1 = \langle -\sin u, \cos u, 0\rangle$$

$$X_v = \langle b\gamma\sinh(\gamma v)\cos u,\ b\gamma\sinh(\gamma v)\sin u,\ 1\rangle$$

$$\frac{X_v}{\| X_v \|} = \frac{\langle \sinh(\gamma v)\cos(u),\ \sinh(\gamma v)\sin(u),\ 1\rangle}{\sqrt{\sinh^2(\gamma v)(\cos^2 u + \sin^2 u) + 1}}$$

But, $\cosh^2\theta - \sinh^2\theta = 1 \quad \therefore \sqrt{1+\sinh^2\theta} = \cosh\theta \geq 1$

Therefore,

$$E_1 = \langle -\sin u, \cos u, 0\rangle$$

$$E_2 = \langle \tanh(v/b)\cos u,\ \tanh(v/b)\sin u,\ \operatorname{sech}(v/b)\rangle$$

$$E_3 = E_1 \times E_2 = \langle \cos u \operatorname{sech}(v/b),\ \sin u \operatorname{sech}(v/b),\ -\tanh(v/b)\rangle$$

(b.) Find $\theta^1, \theta^2, \theta^3$ remember $E_i = \sum_j A_{ij} U_j \leftrightarrow \theta^i = \sum_j A_{ij}\, dx^j$

$$\theta^1 = -\sin u\, dx + \cos u\, dy$$

$$\theta^2 = \tanh(v/b)\cos(u)\, dx + \tanh(v/b)\sin(u)\, dy - \operatorname{sech}(v/b)\, dz$$

$$\theta^3 = \frac{\cos u}{\cosh(v/b)}\, dx + \frac{\sin u}{\cosh(v/b)}\, dy - \frac{\sinh(v/b)}{\cosh(v/b)}\, dz$$

P95 continued

$$dx = b\gamma \sinh(\gamma v)\cos(u)\,dv - b\cosh(\gamma v)\sin(u)\,du$$

$$dy = b\gamma \sinh(\gamma v)\sin(u)\,dv + b\cosh(\gamma v)\cos(u)\,du$$

$$dz = dv$$

Substitute these into θ^3,

$$\theta^3 = \frac{\cos(u)\left[\sinh(\gamma v)\cos u\,dv - b\cancel{\cosh(\gamma v)\sin(u)}\,du\right]}{\cosh(\gamma v)} + $$

$$+ \frac{\sin(u)\left[\sinh(\gamma v)\sin(u)\,dv + b\cancel{\cosh(\gamma v)}\cos(u)\,du\right]}{\cosh(\gamma v)} - \frac{dv\sinh\gamma v}{\cosh(\gamma v)}$$

$$= \frac{\sinh(\gamma v)\,dv - dv\cdot\sinh(\gamma v)}{\cosh(\gamma v)}$$

$$= \left(\frac{\sinh(\gamma v) - \sinh(\gamma v)}{\cosh(\gamma v)}\right)dv$$

$$= 0$$

P95

(c.)

$$E_1 = \frac{X_u}{\|X_u\|} = \frac{1}{b\cosh(v/b)} \frac{\partial}{\partial u}$$

$$E_2 = \frac{X_v}{\|X_v\|} = \frac{1}{\cosh(v/b)} \frac{\partial}{\partial v}$$

$$\theta^1 = b\cosh(v/b)\, du$$

$$\theta^2 = \cosh(v/b)\, dv$$

$$d\theta^1 = \sinh(v/b)\, dv \wedge du = \omega_{12} \wedge \theta^2$$

$$d\theta^2 = \underbrace{0 = -\omega_{12} \wedge \theta^1}_{\nexists \neq 0 \ \theta^2 \text{ comp. of } \omega_{12}}$$

$$\sinh(v/b)\, dv \wedge du = \cosh(v/b)\, \omega_{12} \wedge dv$$

$$\parallel$$

$$-\sinh(v/b)\, du \wedge dv$$

$$\cosh(v/b)\, \omega_{12} = -\sinh(v/b)\, du$$

$$\omega_{12} = -\tanh(v/b)\, du$$

$$= -\tanh(v/b) \frac{\theta^1}{b\cosh(v/b)}$$

$$= \frac{-\sinh(v/b)}{b\cosh^2(v/b)} \theta^1$$

P95 Calculate $d\omega_{12} = -K\theta^1 \wedge \theta^2$

(d.) $$d\omega_{12} = -d(\tanh(v/b)) \wedge du$$

$$= -\frac{\operatorname{sech}^2(v/b)}{b} dv \wedge du$$

$$= \frac{1}{b\cosh^2(v/b)} du \wedge dv$$

$$= \frac{1}{b\cosh^2(v/b)} \left(\frac{\theta^1}{b\cosh(v/b)}\right) \wedge \left(\frac{\theta^2}{\cosh(v/b)}\right)$$

$$= \frac{1}{b^2\cosh^4(v/b)} \theta^1 \wedge \theta^2 = -K\theta^1 \wedge \theta^2$$

$$\therefore \quad \boxed{K = \frac{-1}{b^2\cosh^4(v/b)}}$$

P96 $F: M \to N$ smooth surface map. For each patch $\mathbf{X}: D \to M$ consider the map $\bar{\mathbf{X}} = F \circ \mathbf{X} : D \to N$. Then F is local isometry iff for each patch $\mathbf{X}$ we have $E = \bar{E}$ and $F = \bar{F}$ and $G = \bar{G}$ where

$E = \partial_u \mathbf{X} \cdot \partial_u \mathbf{X}$ and $F = \partial_u \mathbf{X} \cdot \partial_v \mathbf{X}$ and

$G = \partial_v \mathbf{X} \cdot \partial_v \mathbf{X}$. Use this $Th^{\underline{m}}$ to find local isometry of the,

(a.) <u>plane and cylinder</u>

$$\mathbf{X}(u,v) = (u, v, 0) \quad \text{and} \quad \mathbf{Y}(u,v) = \left(R\cos(u/R),\, R\sin(u/R),\, v\right)$$

Following Lecture 20, p. 6 (Example 4b from Oneill)

$$F(u,v,0) = \left(R\cos(u/R),\, R\sin(u/R),\, v\right)$$

$$\Rightarrow \boxed{F(x,y) = \left(R\cos(x/R),\, R\sin(x/R),\, y\right)}.$$

(b.) <u>helicoid and catenoid</u>

$$\mathbf{X}(u,v) = (u\cos v,\, u\sin v,\, v)$$
$$\mathbf{Y}(u,v) = (g,\, h\cos v,\, h\sin v)$$

$g(u) = \sinh^{-1}(u)$

$h(u) = \sqrt{1+u^2}$

$$F(\underbrace{u\cos v}_{x},\, \underbrace{u\sin v}_{y},\, \underbrace{v}_{z}) = \left(\sinh^{-1}(u),\, \sqrt{1+u^2}\cos v,\, \sqrt{1+u^2}\sin v\right)$$

$\tan v = y/x$ and $x^2+y^2 = u^2 \Rightarrow u = \pm\sqrt{x^2+y^2}$

$(\tan z = y/x$ gives Helicoid$)$ $\qquad v = z$

Thus,

$$\boxed{F(x,y,z) = \left(\sinh^{-1}\left(\pm\sqrt{x^2+y^2}\right),\, \cos(z)\sqrt{1+x^2+y^2},\, \sin(z)\sqrt{1+x^2+y^2}\right)}.$$

P97 see Lecture 20, p. 9-13 basically.

P98 $\quad g = \dfrac{4}{(1-x^2-y^2)^2}(dx^2+dy^2)$

has $\quad E_i = \frac{1}{2}(1-x^2-y^2)\dfrac{\partial}{\partial x^i} \quad$ has $\quad g(E_i, E_j) = \delta_{ij}$

and $\quad \theta^1 = \dfrac{2\,dx}{1-x^2-y^2} \qquad \theta^2 = \dfrac{2\,dy}{1-x^2-y^2}$

Calculate K given $M = \{ (x^1, x^2) \mid \|(x^1,x^2)\| < 1 \}$

$$d\theta^1 = \frac{\partial}{\partial y}\left[\frac{2}{1-x^2-y^2}\right] dy \wedge dx = \frac{-2(-2y)}{(1-x^2-y^2)^2}\, dy \wedge dx$$

$$d\theta^1 = \frac{-4y\, dx \wedge dy}{(1-x^2-y^2)^2}$$

$$d\theta^2 = \frac{-2(-2x)\, dx \wedge dy}{(1-x^2-y^2)^2} = \frac{4x\, dx \wedge dy}{(1-x^2-y^2)^2}$$

$$d\theta^1 = \omega_{12} \wedge \theta^2 \longrightarrow \frac{-4y\, dx \wedge dy}{(1-x^2-y^2)^2} = \omega_{12} \wedge \left(\frac{2\,dy}{1-x^2-y^2}\right)$$

$$\therefore \quad \omega_{12}[\partial_x] = \frac{-2y}{(1-x^2-y^2)^2} \quad (1)$$

$$d\theta^2 = -\omega_{12} \wedge \theta^1 \longrightarrow \frac{4x\, dx \wedge dy}{(1-x^2-y^2)^2} = -\omega_{12} \wedge \left(\frac{2\,dx}{1-x^2-y^2}\right)$$

$$\therefore \quad \omega_{12}[\partial_y] = \frac{2x}{(1-x^2-y^2)^2} \quad (2)$$

From (1) and (2)

$$\omega_{12} = \frac{-2y\,dx + 2x\,dy}{(1-x^2-y^2)^2}$$

p98 continued $\quad \alpha = 1 - x^2 - y^2$

$$d\omega_{12} = \frac{\partial}{\partial y}\left[\frac{-2y}{\alpha^2}\right] dy \wedge dx + \frac{\partial}{\partial x}\left[\frac{2x}{\alpha^2}\right] dx \wedge dy$$

$$= \left[\frac{-2}{\alpha^2} - 2y\left(\frac{-2(-2y)}{\alpha^3}\right)\right] dy \wedge dx + \left[\frac{2}{\alpha^2} + 2x\left(\frac{-2(-2x)}{\alpha^3}\right)\right] dx \wedge dy$$

$$= \frac{-2}{\alpha^3}\left(1 - x^2 - y^2 + 4y^2\right) dy \wedge dx + \frac{2}{\alpha^3}\left(1 - x^2 - y^2 + 4x^2\right) dx \wedge dy$$

$$= \frac{2}{\alpha^3}\left[\left(1 - x^2 - y^2 + 4y^2 + 1 - x^2 - y^2 + 4x^2\right)\right] dx \wedge dy$$

$$= \frac{2}{\alpha^3}\left(2 - 2x^2 - 2y^2\right) dx \wedge dy$$

$$= \frac{4}{(1 - x^2 - y^2)^2} dx \wedge dy \quad ①$$

$$= -K\, \theta^1 \wedge \theta^2$$

$$= -K \left(\frac{2\, dx}{1 - x^2 - y^2}\right) \wedge \left(\frac{2\, dy}{1 - x^2 - y^2}\right)$$

$$= \frac{-4K}{(1 - x^2 - y^2)^2} dx \wedge dy \quad ②$$

Compare ① and ② to derive $\boxed{K = -1}$

Remark: $\omega_{12} = \dfrac{2(x^2 + y^2)}{(1 - x^2 - y^2)^2}\left(\dfrac{-y\,dx + x\,dy}{x^2 + y^2}\right) = \dfrac{2(x^2 + y^2)}{(1 - x^2 - y^2)^2}\, d\theta$

$\boxed{\text{p99}}$ To find a metric on the plane with curvature $K=1$ I'll pull-back the natural metric on the unit-sphere via an - the stereographic projection

$\mathbb{R}^3$ Euclidean metric $ds^2 = dx^2 + dy^2 + dz^2$

Euclidean metric in spherical coordinates $ds^2 = d\rho^2 + \rho^2 d\phi^2 + \rho^2 \sin^2\phi \, d\theta^2$

$\Rightarrow$ metric on $\rho = 1$ given by $\underline{ds^2 = d\phi^2 + \sin^2\phi \, d\theta^2}$.

<u>Stereographic Projection</u>

$$(x,y,z) \xmapsto{F} \left(\frac{x}{1-z}, \frac{y}{1-z}\right) = (u,v)$$

$$(u,v) \xmapsto{H=F^{-1}} \left(\underbrace{\frac{2u}{1+u^2+v^2}}_{x}, \underbrace{\frac{2v}{1+u^2+v^2}}_{y}, \underbrace{\frac{-1+u^2+v^2}{1+u^2+v^2}}_{z}\right)$$

$$H : \mathbb{R}^2 \longrightarrow \Sigma = S^2 \subseteq \mathbb{R}^3$$

$$g = H^*\bar{g} = H^*(dx^2 + dy^2 + dz^2) \quad \alpha \overset{\text{def}^{n}}{\equiv} 1 + u^2 + v^2$$

$$= \left(d\left(\frac{2u}{\alpha}\right)\right)^2 + \left(d\left(\frac{2v}{\alpha}\right)\right)^2 + \left(d\left(\frac{u^2+v^2-1}{\alpha}\right)\right)^2$$

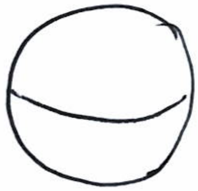
$$= \left(\left(\frac{2}{\alpha} - \frac{4u^2}{\alpha^2}\right)du - \frac{4uv}{\alpha^2}dv\right)^2$$

$$+ \left(-\frac{4uv}{\alpha^2}du + \left(\frac{2}{\alpha} - \frac{4v^2}{\alpha^2}\right)dv\right)^2$$

$$+ \left(\left(\frac{2u}{\alpha} - \frac{2u(u^2+v^2-1)}{\alpha^2}\right)du + \left(\frac{2v}{\alpha} - \frac{2v(u^2+v^2-1)}{\alpha^2}\right)dv\right)^2$$

$(\mathbb{R}^2, g)$ gives a geometry with $K = 1$, forgive me if I skip the verification here, I'd like to find a nicer model for $K=1$ plane...

$\boxed{P100}$ flat sphere

Σ  $\xrightarrow{\quad F \quad}$ $\boxed{\mathbb{R}^2}$ $\quad \bar{g} = du^2 + dv^2$

$\uparrow$
unit sphere

$$F(x,y,z) = \Big(\underbrace{\frac{x}{1-z}}_{u} \, , \, \underbrace{\frac{y}{1-z}}_{v} \Big)$$

$$F^*\bar{g} = d\left(\frac{x}{1-z}\right)^2 + d\left(\frac{y}{1-z}\right)^2$$

$$= \left[\frac{1}{1-z}dx + \frac{x}{(1-z)^2}dz\right]^2 + \left[\frac{1}{1-z}dy + \frac{y}{(1-z)^2}dz\right]^2$$

$$= \frac{1}{(1-z)^4}\Big[(1-z)dx + xdz)^2 + ((1-z)dy + ydz)^2 \Big]$$

$$= \frac{1}{(1-z)^4}\Big[(1-z)^2 dx^2 + 2(1-z)x\,dx\,dz + x^2dz^2 + (1-z)^2dy^2 + 2(1-z)y\,dy\,dz + y^2dz^2 \Big]$$

Alternatively, using usual math sphericals on Σ: $\rho = 1$,

$$F(\phi, \theta) = \Big(\underbrace{\frac{\cos\theta\sin\phi}{1-\cos\phi}}_{u} \, , \, \underbrace{\frac{\sin\theta\sin\phi}{1-\cos\phi}}_{v} \Big)$$

$$du = \left(\frac{-\sin\theta\sin\phi}{1-\cos\phi}\right)d\theta + \left(\frac{\cos\theta\cos\phi}{1-\cos\phi} - \frac{\cos\theta\sin^2\phi}{(1-\cos\phi)^2}\right)d\phi$$

$$dv = \left(\frac{\cos\theta\sin\phi}{1-\cos\phi}\right)d\theta + \left(\frac{\sin\theta\cos\phi}{1-\cos\phi} - \frac{\sin\theta\sin^2\phi}{(1-\cos\phi)^2}\right)d\phi$$

P100 continued

$$du = \frac{1}{1-\cos\phi}\left(-\sin\theta\sin\phi\, d\theta - \cos\theta\, d\phi\right)$$

$$dv = \frac{1}{1-\cos\phi}\left(\cos\theta\sin\phi\, d\theta - \sin\theta\, d\phi\right)$$

$$du^2 + dv^2 = \frac{1}{(1-\cos\phi)^2}\Big[(\sin^2\theta + \cos^2\theta)\sin^2\phi\, d\theta^2$$

$$+ \sin\phi\,(\sin^2\theta - \cos^2\theta)\, d\theta d\phi$$

$$+ (\cos^2\theta + \sin^2\theta)\, d\phi^2\Big]$$

$$= \frac{\sin^2\phi\, d\theta^2 + \sin\phi\,(\sin^2\theta - \cos^2\theta)\, d\theta d\phi + d\phi^2}{(1-\cos\phi)^2}$$

Remark: I wouldn't be too surprised if $\exists$ an error in ↑

Another approach we could try is to use Cor 2.3 from Chapter 7 of Oneill for ruler fnct h,

$$K = h^2 \Delta \log(h) \quad \text{if} \quad \langle v, w\rangle_p = \frac{1}{(h(p))^2}\, v \cdot w$$

Can we solve

$$h^2 (\partial_x^2 + \partial_y^2)(\log(h)) = 1 \qquad (\text{for } P99)$$

$$(\partial_x^2 + \partial_y^2)\log(h) = \frac{1}{h^2}$$

Ansatz, $h = r^\alpha$ where $r = \sqrt{x^2+y^2} \rightsquigarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \ \frac{\partial r}{\partial y} = \frac{y}{r}$

Remark continued

$$\left(\partial_x^2 + \partial_y^2\right) \log\left(r^\alpha\right) = \frac{1}{r^{2\alpha}}$$

$$\left(\partial_x^2 + \partial_y^2\right) \alpha \log(r) = \frac{1}{r^{2\alpha}}$$

$$\partial_x\left(\frac{\alpha}{r}\frac{x}{r}\right) + \partial_y\left(\frac{\alpha}{r}\frac{y}{r}\right) = \frac{1}{r^{2\alpha}}$$

$$\partial_x\left(\frac{\alpha x}{x^2+y^2}\right) + \partial_y\left(\frac{\alpha y}{x^2+y^2}\right) = \frac{1}{r^{2\alpha}}$$

$$\frac{\alpha(x^2+y^2) - \alpha x(2x) + \alpha(x^2+y^2) - \alpha y(2y)}{(x^2+y^2)^2} = \frac{1}{r^{2\alpha}}$$

$$\frac{2\alpha(x^2+y^2) - 2\alpha(x^2+y^2)}{(x^2+y^2)^2} = \frac{1}{r^{2\alpha}}$$

$$0 = \frac{1}{r^{2\alpha}}$$

$\Rightarrow$ not possible to solve via $h = r^\alpha$

Remark continued, can we find h to solve ?

$$K = -\left(\frac{\partial h}{\partial x}\right)^2 - \left(\frac{\partial h}{\partial y}\right)^2 + h\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right) = 1$$

what about

$$h^2 \Delta \log(h) = 1$$

But use $\Delta = \partial_r^2 + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$

I'll aim for $h = g(r)$ thus $\Delta h = g'' + \frac{1}{r} g'$

Solve $g^2(g'' + \frac{1}{r} g') = 1$

$$\Rightarrow 2gg'(g'' + \frac{1}{r}g') + g^2(g''' - \frac{1}{r^2}g' + \frac{1}{r}g'') = 0$$

troublesome.

$\boxed{\text{P101}}$ Gauss Bonnet Thm,

$$\underbrace{\int_M K\,dM}_{\text{Integral of Gaussian curvature over surface oriented by } dM} = 2\pi\, \underbrace{\chi(M)}_{\text{Euler Characteristic of } M}$$

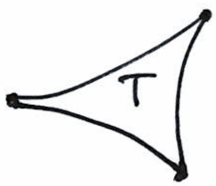
Integral of Gaussian curvature over surface oriented by dM

Euler Characteristic of M

$$\chi(M) = v - e + f$$

v: # of vertices, e: # of edges, f: # of faces

(for a triangularization or rectangularization etc... of M)



$v = 3$
$e = 3$
$f = 1$

$$\chi(T) = 3 - 3 + 1$$

$$\int K\,dM = 2\pi$$, but more to the point Cor 7.6 from Oneill,

$$\int_T K\,dM + \int_{\partial T} \kappa_g\,ds = i_1 + i_2 + i_3 - \pi$$

If ∂T is made of geodesics ($\kappa_g = 0$ for such curves) then

$$\int_T K\,dM = i_1 + i_2 + i_3 - \pi$$

$$\Rightarrow K\int_T dM = K\,\text{area}(T) = i_1 + i_2 + i_3 - \pi.$$

(Let $\text{area}(T) = A$)

$$\boxed{\begin{array}{l} (1.)\ \text{If } K = 0 \text{ then } i_1 + i_2 + i_3 = \pi \\ (2.)\ \text{If } K < 0 \text{ then } i_1 + i_2 + i_3 = \pi + KA < \pi \\ (3.)\ \text{If } K > 0 \text{ then } i_1 + i_2 + i_3 = \pi + KA > \pi \end{array}}$$

$\boxed{P102}$ Explain why the sphere is not isometric to the torus

If $F : \Sigma \longrightarrow \mathbb{T}$ is an isometry then if we have metric g on Σ and $\tilde{g}$ on $\mathbb{T}$ then $F^* \tilde{g} = g$. But, this implies K_Σ and $K_\mathbb{T}$ have related structure,

$$\underbrace{K_{\mathbb{T}}(F(p))}_{\text{both positive and negative}} = \underbrace{K_\Sigma(p)}_{\text{constant value } 1.}$$

Alternatively, F is also homeomorphism and $\chi(\Sigma) = \chi(\mathbb{T})$ which is false.

$$\underset{2}{\chi(\Sigma)} \neq \underset{0}{\chi(\mathbb{T})}$$

P103 Ellipsoid $\mathcal{E} = \{ (x, y, z) \mid x^2/a^2 + y^2/b^2 + z^2/c^2 = 1 \}$ is mapped to the unit-sphere Σ by the map $F : \Sigma \longrightarrow \mathcal{E}$ defined by $F(u, v, w) = (au, bv, cw)$. Explain why F is diffeomorphism, <u>find total curvature of $\mathcal{E}$</u> via GB Th$^{\underline{m}}$ and fact that diffeomorphisms preserve Euler Characteristic.

Note $F^{-1}(x, y, z) = (x/a, y/b, z/c)$ serves as inverce and $J_F = \begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix}$ and $J_{F^{-1}} = \begin{bmatrix} 1/a & & \\ & 1/b & \\ & & 1/c \end{bmatrix}$

thus F and F^{-1} are both smooth $\hookrightarrow$ F is diffeomorphism.

$$\int_{\mathcal{E}} K \, dM = 2\pi \chi(\mathcal{E}) = 2\pi \chi(\Sigma) = \boxed{4\pi}$$