Problems are typically taken from either Jeffrey Lee's text *Manifolds and Differential Geometry* (MDG) or John Lee's text *Smooth Manifolds* (SM). I've also written a few problems. Most problems 5pts here.

Problem 57 Let S be a mixed tensor field on a smooth manifold M such that $S_p : T_pM \times T_pM \times T_p^*M \to \mathbb{R}$ at each $p \in M$. If coordinate charts x and y have domains which overlap then express S in both coordinate charts and show how the components in x and y are related.

Problem 58 Let $T = (x^2 + y^2)dx \otimes dx + xydx \otimes dy$ define a tensor on $M = (0, \infty)^2$. Let $F(u, v, w) = (u + v, v + w)$ define a map from $(0, \infty)^3$ to $(0, \infty)^2$. Calculate the pull-back of T under F; that is, calculate F^*T.

Problem 59 (10pts) Let M be a smooth manifold and suppose $\mathcal{B}$ is the set of rank two smooth covariant tensor fields on M. In particular, if $T \in \mathcal{B}$ then $T_p : T_pM \times T_pM \to \mathbb{R}$ is a bilinear map and $T(X, Y) \in C^\infty(M)$ whenever $X, Y \in \mathfrak{X}(M)$. Prove the following assertions about $\mathcal{B}$:

(a.) any $T \in \mathcal{B}$ can be decomposed into the sum of a symmetric and antisymmetric tensor. Let us denote $\mathcal{B}_A$ for the antisymmetric tensors and $\mathcal{B}_S$ for the symmetric tensors in $\mathcal{B}$.

(b.) if $dx^i \wedge dx^j = dx^i \otimes dx^j - dx^j \otimes dx^i$ then $\gamma_A = \{dx^i \wedge dx^j \mid 1 \leq i < j \leq n\}$ is a point-wise basis for $\mathcal{B}_A$ on the domain of the chart x for M where $dim(M) = n$.

(c.) if $dx^i dx^j = dx^i \otimes dx^j + dx^j \otimes dx^i$ then $\gamma_S = \{dx^i dx^j \mid 1 \leq i \leq j \leq n\}$ is a is a point-wise basis for $\mathcal{B}_S$ on the domain of the chart x for M where $dim(M) = n$,

(d.) comment on the notations for the objects in this problem in terms of terminology from Chapter 12 of John Lee's text.

Problem 60 Suppose $T = \displaystyle\sum_{i,j=1}^{n} T_{ij} dx^i \otimes dx^j$ and suppose T is antisymmetric. If $T = \displaystyle\sum_{i<j} C_{ij} dx^i \wedge dx^j$ then how are the tensor components T_{ij} related to the form components C_{ij} ? Also, how would you define B_{ij} for which $T = \displaystyle\sum_{i,j=1}^{n} B_{ij} dx^i \wedge dx^j$?

Problem 61 SM Exercise 12.3 page 306. (establish property of $F \otimes G$ in the context of multilinear maps)

Problem 62 SM Exercise 12.15 page 315 (prove basic properties of the symmetric product)

Problem 63 SM Exercise 12.26, just part (b), page 320 (prove property of pull-back with respect to $\otimes$ of covariant tensor fields)

Problem 64 SM Problem 12-7 page 325 (breakdown of general covariant tensor into symmetric and antisymmetric is not possible)

Problem 65 SM Exercise 13.13 page 332. (on flatness)

Problem 66 SM Exercise 13.24 page 337. (on curve length)

Problem 67 SM Problem 13-7 page 345. (product of flat metrics flat)

Problem 68 SM Problem 13-7 page 345. (flat $\mathbb{T}^n$)

SOLUTION TO MISSION 6

P57 $S_p : T_pM \times T_pM \times T_p^*M \longrightarrow \mathbb{R}$ a smooth tensor field eval. at $p \in M$. Suppose x and y are coord. charts which overlap at p. Denote $\frac{\partial}{\partial x^i}\Big|_p = \partial_i$ and dx^j for dx^j_p (omit p for brevity) then

$$S(v,w,\alpha) = S\Big(\sum_i v^i \partial_i , \sum_j w^j \partial_j , \sum_k \alpha_k dx^k\Big)$$

$$= \sum_{i,j,k} v^i w^j \alpha_k \underbrace{S(\partial_i, \partial_j, dx^k)}_{S_{ij}{}^k(x)} \quad \underline{*}$$

Likewise, let $y = \bar{x}$ for convenience,

$$S(v,w,\alpha) = \sum_{i,j,k} \bar{v}^i \bar{w}^j \bar{\alpha}_k \underbrace{S(\bar{\partial}_i, \bar{\partial}_j, d\bar{x}^k)}_{\bar{S}_{ij}{}^k(y)} \quad \underline{**}$$

Compare components of S with respect to x and $y = \bar{x}$ by utilizing the chain-rules for $\frac{\partial}{\partial x^i}$ and dx^i

$$\frac{\partial}{\partial \bar{x}^i} = \sum_l \frac{\partial x^l}{\partial \bar{x}^i} \frac{\partial}{\partial x^l}$$

$$d\bar{x}^j = \sum_m \frac{\partial \bar{x}^j}{\partial x^m} dx^m$$

Using $*$ and $**$ and the rules above, multilinearity of S,

$$\boxed{\bar{S}_{ij}{}^k = \sum_{l,m,n} \frac{\partial x^l}{\partial \bar{x}^i} \frac{\partial x^m}{\partial \bar{x}^j} \frac{\partial \bar{x}^k}{\partial x^n} S_{lm}{}^n}$$

PS8 $T = (x^2+y^2)\, dx \otimes dx + xy\, dx \otimes dy$ on $(0,\infty)^2$

and $F(u,v,w) = (u+v,\ v+w)$ maps $(0,\infty)^3$ to $(0,\infty)^2$

We calculate F^*T by setting $x = u+v$ and $y = v+w$ in the formula for T

$$F^*T = (x^2+y^2)\left[(du+dv)\otimes(du+dv)\right] + xy\left[(du+dv)\otimes(dv+dw)\right]$$

$$= (x^2+y^2)\left[du\otimes du + \underline{du\otimes dv} + dv\otimes du + \underline{dv\otimes dv}\right] + xy\left[\underline{du\otimes dv} + du\otimes dw + \underline{dv\otimes dv} + dv\otimes dw\right]$$

$$\begin{aligned}
F^*T &= \left[(u+v)^2 + (v+w)^2\right] du\otimes du \\
&+ \left[(u+v)^2 + (v+w)^2 + (u+v)(v+w)\right] du\otimes dv \\
&+ \left[(u+v)^2 + (v+w)^2\right] dv\otimes du \\
&+ \left[(u+v)^2 + (v+w)^2 + (u+v)(v+w)\right] dv\otimes dv \\
&+ \left[(u+v)(v+w)\right] du\otimes dw \\
&+ \left[(u+v)(v+w)\right] dv\otimes dw
\end{aligned}$$

yep.

$\boxed{P59}$ $\mathcal{B}$ set of rank two smooth covariant tensor fields on M.

(a.) $$T(V,W) = \underbrace{\tfrac{1}{2}\big(T(V,W) + T(W,V)\big)}_{T_S(V,W)} + \underbrace{\tfrac{1}{2}\big(T(V,W) - T(W,V)\big)}_{T_A(V,W)}$$

Observe $T_S(V,W) = T_S(W,V)$ and $T_A(V,W) = -T_A(W,V)$ thus $T_S \in \mathcal{B}_S$ and $T_A \in \mathcal{B}_A$ as bilinearity of T_S, T_A is easy to check.

(b.) If $T \in \mathcal{B}_A$ then $T(V,W) = -T(W,V)$ thus

$$\begin{aligned}
T &= \sum_{i,j} T_{ij}\, dx^i \otimes dx^j \qquad T_{ij} = T(\partial_i, \partial_j) \\
& \qquad\qquad\qquad\qquad\quad T_{ii} = 0 \text{ by antisymm.} \\
&= \sum_{i<j} T_{ij}\, dx^i \otimes dx^j + \sum_{i>j} T_{ij}\, dx^i \otimes dx^j \\
&= \sum_{k<l} T_{kl}\, dx^k \otimes dx^l + \sum_{l>k} T_{lk}\, dx^l \otimes dx^k \qquad T_{lk} = -T_{kl} \\
&= \sum_{k<l} T_{kl}\left(dx^k \otimes dx^l - dx^l \otimes dx^k\right) \\
&= \sum_{k<l} T_{kl}\, dx^k \wedge dx^l \quad \Rightarrow \gamma_A \text{ spans } \mathcal{B}_A
\end{aligned}$$

Linear independence of $\gamma_A = \{dx^i \wedge dx^j \mid 1 \le i < j \le n\}$ follows from $T = 0 \Rightarrow T_{kl} = 0$ for the eq$^{\underline{n}}$ above.

P59 continued

(c.) generally $T = \sum_{i,j}' T_{ij}\, dx^i \otimes dx^j$ for T a covariant rank two tensor on M. If $T \in \mathcal{B}_S$ then $T(v,w) = T(w,v)$ and thus

$$T_{ij} = T\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = T\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\right) = T_{ji} \quad \forall i,j$$

Therefore,

$$\begin{aligned}
T &= \sum_{k,l} T_{kl}\, dx^k \otimes dx^l \\
&= \sum_{k<l} T_{kl}\, dx^k \otimes dx^l + \sum_{k>l}' T_{kl}\, dx^k \otimes dx^l + \sum_{k}' T_{kk}\, dx^k \otimes dx^k \\
&= \sum_{i<j}' T_{ij}\, dx^i \otimes dx^j + \sum_{j>i}' T_{ji}\, dx^j \otimes dx^i + \sum_{k}' T_{kk}\, dx^k \otimes dx^k \\
&= \sum_{i<j}' T_{ij}\, (dx^i \otimes dx^j + dx^j \otimes dx^i) + \sum_{i}' T_{ii}\, dx^i \otimes dx^i \\
&= \sum_{i<j}' T_{ij}\, dx^i dx^j + \sum_{i=j}' T_{ij}\, dx^i dx^j \\
&= \sum_{i \le j}' T_{ij}\, dx^i dx^j \quad \Rightarrow \quad \gamma_S = \{ dx^i dx^j \mid 1 \le i \le j \le n \}
\end{aligned}$$

gives point-wise basis for $\mathcal{B}_S$

(d.) $\mathcal{T}^2(M) = \mathcal{B} = \Gamma(T^2 T^* M)$ (Chapter 12)

$\Omega^2(M) = \mathcal{B}_A = \Gamma(\Lambda^2 T^* M)$ (Chapter 14, p. 360)

(John Lee's Smooth Manifolds)

P60 Given $T = \sum_{i,j}' T_{ij}\, dx^i \otimes dx^j$ and $T \in \mathcal{B}_A$

If $T = \sum_{i<j} C_{ij}\, dx^i \wedge dx^j$ then how are T_{ij} & C_{ij} related.

Also, how to define B_{ij} for which $T = \sum_{i,j}' B_{ij}\, dx^i \wedge dx^j$?

By Problem 59b we've already shown $T_{ij} = C_{ij}$ for $i<j$.

and of course $T_{ij} = -T_{ji} = C_{ij} \Rightarrow T_{ji} = -C_{ij}$ for $j>i$

If $T = \sum_{i,j} B_{ij}\, dx^i \wedge dx^j$ where $dx^i \wedge dx^j = dx^i \otimes dx^j - dx^j \otimes dx^i$,

$$T = \sum_{i,j}' B_{ij}\left(dx^i \otimes dx^j - dx^j \otimes dx^i\right)$$

could make B_{ii} whatever.

Ok, I've got another idea,

$$T = \sum_{i<j}' B_{ij}\, dx^i \wedge dx^j + \sum_{i>j}' B_{ij}\, dx^i \wedge dx^j + \overbrace{\sum_{i=j} B_{ii}\, \underbrace{dx^i \wedge dx^i}_{0}}$$

$$= \sum_{i<j}' B_{ij}\, dx^i \wedge dx^j + \sum_{i>j}' -B_{ij}\, dx^j \wedge dx^i$$

$$= \sum_{k<l}' B_{kl}\, dx^k \wedge dx^l + \sum_{l>k}' -B_{lk}\, dx^k \wedge dx^l$$

$$= \sum_{k<l}' \left(B_{kl} - B_{lk}\right) dx^k \wedge dx^l = \sum_{k<l}' T_{kl}\, dx^k \wedge dx^l$$

We need to define B_{ij} s.t. $T_{kl} = B_{kl} - B_{lk}$ for $k<l$. If we suppose $B_{lk} = -B_{kl}$ for all l,k then

$$T_{kl} = B_{kl} - (-B_{kl}) = 2B_{kl} \Rightarrow \boxed{B_{ij} = \frac{1}{2} T_{ij}}$$

(this answer is not unique, but it's usually what's done)

P61 §12.3 Ex. 12.3 p.306

$F \in L(V_1, \dots, V_k)$ $G \in L(W_1, \dots, W_\ell)$ $H \in L(U_1, \dots, U_m)$

I'll just prove associativity $(F \otimes G) \otimes H = F \otimes (G \otimes H)$.

Let $x_i \in V_i$ and $y_j \in W_j$ and $z_b \in U_b$ for $i = 1, 2, \dots, k$ and $j = 1, 2, \dots, l$ and $b = 1, 2, \dots, m$. Consider,

$\vec{x} = (x_1, \dots, x_k)$
$\vec{y} = (y_1, \dots, y_l)$
$\vec{z} = (z_1, \dots, z_b)$

$$\begin{aligned}
\big((F \otimes G) \otimes H\big)(\vec{x}, \vec{y}, \vec{z}) &= (F \otimes G)(\vec{x}, \vec{y})\, H(\vec{z}) \\
&= \big(F(\vec{x})\, G(\vec{y})\big) H(\vec{z}) \\
&= F(\vec{x}) \big(G(\vec{y})\, H(\vec{z})\big) \quad \text{associativity of mult. in } \mathbb{R} \\
&= F(\vec{x}) (G \otimes H)(\vec{y}, \vec{z}) \\
&= \big(F \otimes (G \otimes H)\big)(\vec{x}, \vec{y}, \vec{z})
\end{aligned}$$

Thus $(F \otimes G) \otimes H = F \otimes (G \otimes H)$.

Fine, I'll do bilinearity, $F_1, F_2 \in L(V_1, \dots, V_k)$

$$\begin{aligned}
\big((F_1 + F_2) \otimes G\big)(\vec{x}, \vec{y}) &= (F_1 + F_2)(\vec{x})\, G(\vec{y}) \quad \text{def}^{\underline{n}} \text{ of } F_1 + F_2 \\
&= \big(F_1(\vec{x}) + F_2(\vec{x})\big) G(\vec{y}) \quad \text{prop. of } \mathbb{R} \text{ addition / mult.} \\
&= F_1(\vec{x}) G(\vec{y}) + F_2(\vec{x}) G(\vec{y}) \\
&= (F_1 \otimes G)(\vec{x}, \vec{y}) + (F_2 \otimes G)(\vec{x}, \vec{y}) \\
&= (F_1 \otimes G + F_2 \otimes G)(\vec{x}, \vec{y})
\end{aligned}$$

Holds $\forall\, \vec{x} \in V_1 \times \cdots \times V_k$ and $\vec{y} \in W_1 \times \cdots \times W_\ell$ hence

$$(F_1 + F_2) \otimes G = F_1 \otimes G + F_2 \otimes G$$

Proof of $F \otimes (G_1 + G_2) = F \otimes G_1 + F \otimes G_2$ and $F \otimes (cG) = c\, F \otimes G$ and $(cF) \otimes G = c(F \otimes G)$ are similar.

$\boxed{P62}$ $\alpha\beta(v_1,\dots,v_{k+l}) = \frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}} \alpha(v_{\sigma(1)},\dots,v_{\sigma(k)})\,\beta(v_{\sigma(k+1)},\dots,v_{\sigma(k+l)})$

"k+l"

Show that,

(a.) $\alpha\beta = \beta\alpha$ and $(a\alpha+b\beta)\gamma = a\alpha\gamma + b\beta\gamma = \gamma(a\alpha+b\beta)$

(b.) for $\alpha,\beta\in V^*$ then $\alpha\beta = \frac{1}{2}(\alpha\otimes\beta + \beta\otimes\alpha)$

$$(\beta\alpha)(\vec{v}) = \frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}} \beta(v_{\sigma(1)},\dots,v_{\sigma(l)})\,\alpha(v_{\sigma(l+1)},\dots,v_{\sigma(l+k)})$$

$$= \frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}} \alpha(\underbrace{v_{\sigma(l+1)},\dots,v_{\sigma(l+k)}}_{\substack{\text{ranges over}\\ \text{all permutations}\\ \text{of } k\text{-vectors}\\ \text{taken from } \vec{v}}})\,\beta(\underbrace{v_{\sigma(1)},\dots,v_{\sigma(l)}}_{\substack{\text{ranges over}\\ \text{all permutations}\\ \text{of } l\text{-vectors}\\ \text{taken from } \vec{v}}})$$

$\vec{v} = (v_1, v_2, \dots, v_{k+l})$

$$= \frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}} \alpha(v_{\sigma(1)},\dots,v_{\sigma(k)})\,\beta(v_{\sigma(k+1)},\dots,v_{\sigma(k+l)})$$

$$= (\alpha\beta)(\vec{v}) \Rightarrow \underline{\alpha\beta = \beta\alpha}.$$

Let $a,b\in\mathbb{R}$ and α,β,γ completely symmetric tensors,

$$(a\alpha+b\beta)\gamma(\vec{v}) = \frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}} (a\alpha+b\beta)(\underbrace{v_{\sigma(1)},\dots,v_{\sigma(k)}}_{x})\,\gamma(\underbrace{v_{\sigma(k+1)},\dots,v_{\sigma(k+l)}}_{y})$$

$$= \frac{a}{(k+l)!}\sum_{\sigma} \alpha(x)\gamma(y) + \frac{b}{(k+l)!}\sum_{\sigma} \beta(x)\gamma(y)$$

$$= (a\alpha\gamma + b\beta\gamma)(\vec{v})$$ (sorry I skipped a couple steps here)

$\therefore (a\alpha+b\beta)\gamma = a\alpha\gamma + b\beta\gamma$

Then the proof of $a\alpha\gamma + b\beta\gamma = \gamma(a\alpha+b\beta)$ is similar.

P62 continued for $\alpha, \beta \in V^*$

$$\alpha\beta(v_1, v_2) = \frac{1}{2!}\sum_{\sigma \in S_2} \alpha(v_{\sigma(1)})\, \beta(v_{\sigma(2)})$$

commute

$$= \frac{1}{2}\left(\alpha(v_1)\beta(v_2) + \alpha(v_2)\beta(v_1)\right)$$

$$= \frac{1}{2}\left((\alpha\otimes\beta)(v_1,v_2) + (\beta\otimes\alpha)(v_1,v_2)\right)$$

$$= \frac{1}{2}(\alpha\otimes\beta + \beta\otimes\alpha)(v_1, v_2)$$

Therefore, $\underline{\alpha\beta = \frac{1}{2}(\alpha\otimes\beta + \beta\otimes\alpha)}$.

P63 Ex. 12.26b, p. 320 /
$F: M \longrightarrow N$ smooth map and A, B smooth covariant tensor fields on N then show $F^*(A\otimes B) = F^*A \otimes F^*B$

Suppose A has rank k and B rank l and suppose $\vec{v} = v_1, \ldots, v_{k+l} \in T_pM$ so $dF_p(v_i) \in T_{F(p)}N \;\forall i$.

$$F^*(A\otimes B)_p(\vec{v}) = (A\otimes B)_{F(p)}\left(dF_p(v_1), \ldots, dF_p(v_{k+l})\right)$$

$$= A_{F(p)}\left(dF_p(v_1), \ldots, dF_p(v_k)\right) B_{F(p)}\left(dF_p(v_{k+1}), \ldots, dF_p(v_{k+l})\right)$$

$$= (F^*A)_p(v_1, \ldots, v_k)\,(F^*B)_p(v_{k+1}, \ldots, v_{k+l})$$

$$= \left((F^*A)_p \otimes (F^*B)_p\right)(\vec{v})$$

Therefore, $\underline{F^*(A\otimes B) = (F^*A)\otimes(F^*B)}$. //

$\boxed{P64}$ Show $e^1 \otimes e^2 \otimes e^3$ is not sum of sym. and antisymmetric tensor over $\mathbb{R}^3$. Prob 12-7, p. 325

Towards $\rightarrow\leftarrow$ suppose S symmetric and A antisymmetric such that $e^1 \otimes e^2 \otimes e^3 = T = S + A$

Notice $(e^1 \otimes e^2 \otimes e^3)(e_i, e_j, e_k) = e^1(e_i)\, e^2(e_j)\, e^3(e_k)$

$$= \delta_{1,i}\, \delta_{2j}\, \delta_{3k} \quad (\star)$$

Thus $T_{ijk} = 1$ only if $i=1$, $j=2$, $k=3$ and else $T_{ijk} = 0$. Since S symmetric we have $S_{ijk} = S_{jik} = S_{ikj}$ etc... Whereas $A_{ijk} = -A_{jik}$ etc. Consider,

$$1 = S_{123} + A_{123} = S_{213} - A_{213}$$

$$0 = S_{213} + A_{213}$$

Thus $1 = 2S_{213}$ adding the eq^{n}'s above.

Hence $S_{213} = \frac{1}{2}$ and so $\underline{S_{123} = \frac{1}{2}}$. $\Rightarrow$ $\underline{A_{123} = \frac{1}{2}}$.

In fact, from $\star$ we find $S_{ijk} = 0$ for any i, j, k for which there is any repeated index. Thus, P64 continued

$$S_{123} = S_{213} = S_{132} = S_{321} = S_{213} = S_{132} = \frac{1}{2}$$

$$A_{123} = A_{231} = A_{312} = -A_{321} = -A_{213} = -A_{132} = \frac{1}{2}$$

Thus, using notation $e^{ijk} = e^i \otimes e^j \otimes e^k$

$$e^1 \otimes e^2 \otimes e^3 = \frac{1}{2}\left(e^{123} + e^{231} + e^{312} + e^{321} + e^{213} + e^{132}\right)$$

$$+ \frac{1}{2}\left(e^{123} + e^{231} + e^{312} - e^{321} - e^{213} - e^{132}\right)$$

$$= e^{123} + e^{231} + e^{312}$$

$$= e^1 \otimes e^2 \otimes e^3 + e^2 \otimes e^3 \otimes e^1 + e^3 \otimes e^1 \otimes e^2$$

$$\Rightarrow \quad e^2 \otimes e^3 \otimes e^1 + e^3 \otimes e^1 \otimes e^2 = 0$$

But evaluation on e_2, e_3, e_1 shows $1 = 0 \;\rightarrow\leftarrow$

Thus $\nexists\, S, A$ s.t. $e^1 \otimes e^2 \otimes e^3 = S + A$ where S symmetric and A antisymmetric.

Remark: there is probably a shorter argument, but this is the one I found today.

$\boxed{\text{PG5}}$ Ex. 13.13 on p-332

Suppose (M, g) and $(M, \tilde{g})$ are isometric Riemannian manifolds. Show g is flat iff $\tilde{g}$ is flat

We say (M, g) is <u>flat</u> if it is locally isometric to Euclidean space $(\mathbb{R}^n, \bar{g})$.

$\Rightarrow$) Suppose g is flat then there exists a local isometry $G : \mathbb{R}^n \longrightarrow M$ and $G^* g = \bar{g}$. Since (M, g) and $(M, \tilde{g})$ are isometric Riemannian manifolds there is also a map $F : M \longrightarrow \tilde{M}$ s.t. $F^* \tilde{g} = g$.

Then $F \circ G : \mathbb{R}^n \longrightarrow \tilde{M}$ has

$$\begin{aligned} (F \circ G)^* \tilde{g} &= G^*(F^* g) \\ &= G^*(g) \\ &= \bar{g} \end{aligned} \quad \Rightarrow (\tilde{M}, \tilde{g}) \text{ locally isometric to } (\mathbb{R}^n, \bar{g}) \therefore \underline{\tilde{g} \text{ flat}}.$$

$\Leftarrow$) If $\tilde{g}$ is flat and (M, g) is isometric to $(\tilde{M}, \tilde{g})$ then there is map $F : M \longrightarrow \tilde{M}$ with $F^* \tilde{g} = g$

But $\tilde{g}$ flat so there is map $H : \tilde{M} \longrightarrow \mathbb{R}^n$ with $H^* \bar{g} = \tilde{g}$

Thus $H \circ F : M \xrightarrow{F} \tilde{M} \xrightarrow{H} \mathbb{R}^n$ has

$$\begin{aligned} (H \circ F)^* \bar{g} &= F^*(H^* \bar{g}) \\ &= F^*(\tilde{g}) \\ &= g \end{aligned} \quad \Rightarrow \underline{g \text{ is flat}}.$$

$\boxed{\text{PGG}}$ Ex. 13.24 p. 337

Suppose (M, g) and $(\tilde{M}, \tilde{g})$ are Riemannian manifolds with or w/o boundary, and $F: M \to \tilde{M}$ is a local isometry. Show $L_{\tilde{g}}(F \circ \gamma) = L_g(\gamma)$ for any piecewise smooth curve segment γ in M

We are given $F^*\tilde{g} = g$. Let $\gamma : [a, b] \to M$ then define $L_g(\gamma) = \int_a^b |\gamma'(t)|_g \, dt$ where

$$|\gamma'(t)|_g = \sqrt{g(\gamma'(t), \gamma'(t))}$$

Consider $F \circ \gamma : [a, b] \longrightarrow \tilde{M}$ is smooth curve seg. in $\tilde{M}$

$$L_{\tilde{g}}(F \circ \gamma) = \int_a^b |(F \circ \gamma)'(t)|_{\tilde{g}} \, dt \qquad (F \circ \gamma)'(t) = dF_{\gamma(t)}(\gamma'(t))$$

$$= \int_a^b \sqrt{\tilde{g}(dF(\gamma'(t)), dF(\gamma'(t)))} \, dt$$

$$= \int_a^b \sqrt{(F^*\tilde{g})(\gamma'(t), \gamma'(t))} \, dt$$

$$= \int_a^b \sqrt{g(\gamma'(t), \gamma'(t))} \, dt$$

$$= \int_a^b |\gamma'(t)|_g \, dt$$

$$= L_g(\gamma)$$

$\hookrightarrow$ length of curve segment is isometric invariant of Riemannian geometry.

$\boxed{P67}$ Problem 13-7, p. 345

Show that the product of flat metrics is flat

Given Riemannian manifolds (M, g) and $(\tilde{M}, \tilde{g})$ we define $\hat{g} = g \oplus \tilde{g}$ on $M \times \tilde{M}$ called the <u>product metric</u> as follows (from p. 329 in Lee)

$$\hat{g}\big((v,\tilde{v}),(w,\tilde{w})\big) = g(v,w) + \tilde{g}(\tilde{v},\tilde{w})$$

Suppose (M, g) and $(\tilde{M}, \tilde{g})$ are flat then I'll use Th$^{\underline{m}}$ 13.14 part (b.) which tells us flatness can be formulated as each pt. allows coordinate system in which the components of the metric are δ_{ij}

Let $(p, q) \in M \times \tilde{M}$ then $\exists$ coord. x about p in M and $\exists$ coord y about q in $\tilde{M}$ for which

$$\left.\begin{aligned} g &= \sum_{i,j=1}^{m} \delta_{ij}\, dx^i dx^j = \sum_{i=1}^{m} dx_i^2 \\ \bar{g} &= \sum_{i,j=1}^{n} \delta_{ij}\, dy^i dy^j = \sum_{i=1}^{n} dy_i^2 \end{aligned}\right\} *$$

Consider then, basis for $T_{(p,q)}(M \times \tilde{M})$ given by $\left\{ \left(\frac{\partial}{\partial x^i}, 0\right) \right\} \cup \left\{ \left(0, \frac{\partial}{\partial y^i}\right) \right\}$,

$$\begin{aligned} \hat{g}\big(\underbrace{(v,\tilde{v}),(w,\tilde{w})}_{\text{☺}}\big) &= g(v,w) + \tilde{g}(\tilde{v},\tilde{w}) \quad \text{using } * \\ &= v^1 w^1 + \cdots + v^m w^m + \tilde{v}^1 \tilde{w}^1 + \cdots + \tilde{v}^n \tilde{w}^n \\ &= \left(dz_1^2 + \cdots + dz_m^2 + dz_{m+1}^2 + \cdots + dz_{m+n}^2\right)(\text{☺}) \end{aligned}$$

where $dz_1, \ldots, dz_m$ are dual to $\{(\partial/\partial x^i, 0)\}$ and $dz_{m+1}, \ldots, dz_{m+n}$ are dual to $\{(0, \partial/\partial y^i)\}$. Thus <u>$\hat{g}$ is flat</u>.

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$$\mathbb{T}^n = S^1 \times \cdots \times S^1 \subseteq \mathbb{C}^n$$

Let g be the metric on $\mathbb{T}^n$ induced from the Euclidean metric on $\mathbb{C}^n = \mathbb{R}^{2n}$. Show g flat.

$$\bar{g} = |dz_1|^2 + |dz_2|^2 + \cdots + |dz_n|^2$$
$$= \underbrace{dx_1^2 + dy_1^2 + dx_2^2 + dy_2^2 + \cdots + dx_n^2 + dy_n^2}_{\text{Euclidean metric on } \mathbb{C}^n}$$

$F : \mathbb{T}^n \longrightarrow \mathbb{C}^n$ defined by

$$F(\theta_1, \ldots, \theta_n) = (e^{i\theta_1}, \ldots, e^{i\theta_n})$$
$$= (\underbrace{\cos\theta_1}_{x_1}, \underbrace{\sin\theta_1}_{y_1}, \ldots, \underbrace{\cos\theta_n}_{x_n}, \underbrace{\sin\theta_n}_{y_n})$$

Notice $d\cos\theta_j = -\sin\theta_j \, d\theta_j$ and $d(\sin\theta_j) = \cos\theta_j \, d\theta_j$

thus $(d\cos\theta_j)^2 + (d\sin\theta_j)^2 = (\sin^2\theta_j + \cos^2\theta_j)\, d\theta_j^2 = d\theta_j^2$

Consequently,

$$F^*\bar{g} = \underbrace{d\theta_1^2 + d\theta_2^2 + \cdots + d\theta_n^2}_{\text{flat metric on } \mathbb{T}^n.}$$