Problems are typically taken from either Jeffrey Lee's text *Manifolds and Differential Geometry* (MDG) or John Lee's text *Smooth Manifolds* (SM). I've also written a few problems. Most problems 5pts here.

Problem 67 SM Exercise 14-17 page 361. (make ample use of theorems for covariant tensor pull-back, this should not be a hard exercise)

Problem 68 SM Exercise 14-31 page 369. (formula connecting a Lie Bracket and an exterior derivative)

Problem 69 SM Problem 14-1 page 373. (wedge product and linear dependence)

Problem 70 SM Problem 14-5 page 373. (Cartan's Lemma)

Problem 71 SM Problem 14-6 page 373. (two-form calculation in $\mathbb{R}^3$)

Problem 72 SM Problem 14-7 page 373. (differential form fun)

Solution to Mission 7

P67 Ex. 14-17, p. 361

$F : M \to N$ smooth

(a.) $F^* : \Omega^k(N) \to \Omega^k(M)$ is linear over $\mathbb{R}$

(b.) $F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta)$

(c.) In any smooth chart,

$$F^*\left(\sum_I{}' \omega_I \, dy^{i_1} \wedge \cdots \wedge dy^{i_k} \right) = \sum_I (\omega_I \circ F)\, d(y^{i_1} \circ F) \wedge \cdots \wedge d(y^{i_k} \circ F)$$

For $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$ we define as in Lee pg. 355,

$$\omega \wedge \eta = \frac{(k+l)!}{k!\,l!} \operatorname{Alt}(\omega \otimes \eta)$$

Then proof of (b.) can be given

$$\begin{aligned}
F^*(\omega \wedge \eta) &= F^*\left(\frac{(k+l)!}{k!\,l!} \operatorname{Alt}(\omega \otimes \eta) \right) && \text{by linearity of } F^* \\
&= \frac{(k+l)!}{k!\,l!} \operatorname{Alt}\left(F^*(\omega \otimes \eta) \right) \\
&= \frac{(k+l)!}{k!\,l!} \operatorname{Alt}\left((F^*\omega) \otimes (F^*\eta) \right) && \text{property off } F^* \\
&= (F^*\omega) \wedge (F^*\eta)
\end{aligned}$$

P67 continued (part a)

Let $a \in \mathbb{R}$ and $\beta, \gamma \in \Omega^k(N)$ let $V_1, .., V_k \in T_{F(P)}M$

$$\left(F^*(a\beta+\gamma)\right)(V_1, .., V_k) =$$

$$= (a\beta+\gamma)(dF(V_1), \dots, dF(V_k))$$

$$= (a\beta)(dF(V_1), .., dF(V_k)) + \gamma(dF(V_1), .., dF(V_k))$$

$$= (aF^*\beta + F^*\gamma)(V_1, .., V_k)$$

Thus $F^*(a\beta+\gamma) = aF^*\beta + F^*\gamma$.

Part c) for $V_1, .., V_k \in T_pM$,

$$F^*\left(\sum_I \omega_I dy^{i_1}\wedge..\wedge dy^{i_k}\right)_p(V_1, \dots, V_k) =$$

$$= \left(\sum_I \omega_I dy^{i_1}\wedge\dots\wedge dy^{i_k}\right)_{F(P)}(dF_p(V_1), \dots, dF_p(V_k))$$

$$= \Lambda \sum_I \omega_I(F(P))\, dy^{i_1}(dF_p(V_1))\, dy^{i_2}(dF_p(V_2)) \cdots dy^{i_k}(dF_p(V_k))$$

$$= \Lambda \sum_I \omega_I(F(P))\, d(y^{i_1}\circ F)(V_1) \cdots d(y^{i_k}\circ F)(V_k)$$

$$= \left(\sum_I \omega_I(F(P))\, d(y^{i_1}\circ F)\wedge\dots\wedge d(y^{i_k}\circ F)\right)(V_1, .., V_k)$$

Then (c.) follows. I've used "Λ" to denote the complete antisymmetrization of the products in question this amounts to details of $\alpha_1\wedge\alpha_2\wedge..\wedge\alpha_k$. Sorry me lazy.

P68 Ex 14-31, p. 369

(Given $d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$)

Let M be smooth n-manifold, and (E_i) a smooth local frame for M and (ε^i) the dual coframe. For each i, let $b^i{}_{jk}$ denote comp. of exterior derivative of ε^i in this frame, and for each j,k let $c^i{}_{jk}$ be comp. fncts of Lie Bracket $[E_j, E_k]$;

$$d\varepsilon^i = \sum_{j<k} b^i{}_{jk}\, \varepsilon^j \wedge \varepsilon^k$$

$$[E_j, E_k] = \sum_i c^i{}_{jk} E_i$$

Then $b^i{}_{jk} = -c^i{}_{jk}$

Let $l<m$ and consider,

$$(d\varepsilon^i)(E_l, E_m) = \sum_{j<k} b^i{}_{jk}\, \underbrace{\varepsilon^j \wedge \varepsilon^k}_{\varepsilon^j\otimes\varepsilon^k - \varepsilon^k\otimes\varepsilon^j}(E_l, E_m)$$

$$= \sum_{j<k} b^i{}_{jk}\, \delta_{jl}\, \delta_{km}$$

$$= b^i{}_{lm}$$

Likewise,

$$d\varepsilon^i(E_l, E_m) = E_l(\underbrace{\varepsilon^i(E_m)}_{\delta_{im}}) - E_m(\underbrace{\varepsilon^i(E_l)}_{\delta_{il}}) - \varepsilon^i([E_l, E_m])$$

$$= -\varepsilon^i\left(\sum_k c^k{}_{lm} E_k\right)$$

$$= -\sum_k c^k{}_{lm}\, \varepsilon^i(E_k)$$

$$= -\sum_k c^k{}_{lm}\, \delta_{ik} = \underline{-c^i{}_{lm}}. //$$

P69 Problem 14-1, p. 373

Show covectors $\omega^1, \dots, \omega^k$ on a finite dimensional vector space are linearly dependent iff $\omega^1 \wedge \cdots \wedge \omega^k = 0$

$\Rightarrow$] Suppose $\omega^j = \sum_{i \neq j} c_i \omega^i$ then consider

$$\omega^1 \wedge \cdots \wedge \omega^k = \omega^1 \wedge \cdots \wedge \left(\sum_{i \neq j} c_i \omega^i \right) \wedge \cdots \wedge \omega^k$$

$$= \sum_{i \neq j} c_i \underbrace{\omega^1 \wedge \cdots \wedge \omega^{j-1} \wedge \omega^i \wedge \omega^{j+1} \wedge \cdots \wedge \omega^k}_{i \in \{1, \dots, j-1, j+1, \dots, k\}}$$

$$= \sum_{i \neq j} \pm c_i \underbrace{\omega^i \wedge \omega^i}_{0} \wedge \underbrace{\omega^1 \wedge \cdots \wedge \omega^k}_{\text{missing } \omega^i \text{ and } \omega^j}$$

$$= 0.$$

$\Leftarrow$] Suppose $\omega^1 \wedge \cdots \wedge \omega^k = 0$

Remark: if $\omega^1, \dots, \omega^k$ are LI then $W^* = \text{span}\{\omega^1, \dots, \omega^k\}$ has basis $\omega^1, \dots, \omega^k$ dual to $\varepsilon_1, \dots, \varepsilon_n$ and thus $(\omega^1 \wedge \cdots \wedge \omega^k)(\varepsilon_1, \dots, \varepsilon_n) = 1 \Rightarrow \omega^1 \wedge \cdots \wedge \omega^k \neq 0 \rightarrow\leftarrow$.

P69 continued

(perhaps the argument already given is better ☺)

$\Leftarrow$ Suppose $\omega^1 \wedge \cdots \wedge \omega^k = 0$

Suppose towards $\rightarrow\leftarrow$ that $\omega^1, \dots, \omega^k$ are LI

Then $c_1\omega^1 + \cdots + c_k\omega^k = 0 \Rightarrow c_1 = c_2 = \cdots = c_k = 0$.

Thus $\underline{[\vec{\omega}^1 | \vec{\omega}^2 | \cdots | \vec{\omega}^k]\vec{c} = 0 \Rightarrow \vec{c} = 0.}$ $*$

where $\vec{\omega}^i$ is the coordinate vector of ω^i with respect to some basis for $\mathrm{span}(\omega^1, \dots, \omega^k)$ then

$$\vec{\omega}^1 \wedge \vec{\omega}^2 \wedge \cdots \wedge \vec{\omega}^k = \underbrace{\det([\vec{\omega}^1 | \vec{\omega}^2 | \cdots | \vec{\omega}^k])}_{\substack{\neq 0 \text{ since } * \\ \text{gives invertibility} \\ \text{of this matrix}}} e_1 \wedge \cdots \wedge e_k$$

Yet,

$\omega^1 \wedge \cdots \wedge \omega^k = 0$

implies (Lemma)

$\vec{\omega}^1 \wedge \cdots \wedge \vec{\omega}^k = 0$ $\quad \rightarrow\leftarrow$ $\quad \therefore\ \omega^1, \dots, \omega^k$ are not LI. That is, $\omega^1, \dots, \omega^k$ are linearly dependent.

Lemma: If $\alpha^1, \dots, \alpha^k$ are covectors with coordinate vectors $\vec{\alpha}^1, \dots, \vec{\alpha}^k$ in the sense $\alpha^i = \sum_j \vec{\alpha}^i_j \theta^j$ and $\alpha^1 \wedge \cdots \wedge \alpha^k = 0$ then $\vec{\alpha}^1 \wedge \cdots \wedge \vec{\alpha}^k \overset{?}{=} 0$.

P69 continued

Lemma: $\alpha^1 \wedge \cdots \wedge \alpha^k = 0 \Rightarrow \vec{\alpha}^1 \wedge \cdots \wedge \vec{\alpha}^k = 0$ where $\vec{\alpha}^i$ is coordinate vector of α^i

Suppose $\alpha^i = \sum_j \vec{\alpha}^i_j \theta^j$ where $\theta^1, \dots, \theta^k$ give basis for span $\{\alpha^1, \dots, \alpha^k\}$ and if needed extend past the span, certainly we can view $\alpha^1, \dots, \alpha^k$ as subset of k-dim'l v-space. We have $\theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^k \neq 0$. Consider

$$\alpha^1 \wedge \cdots \wedge \alpha^k = \Big(\sum_{i_1} \vec{\alpha}^1_{i_1} \theta^{i_1}\Big) \wedge \cdots \wedge \Big(\sum_{i_k} \vec{\alpha}^k_{i_k} \theta^{i_k}\Big)$$

$$= \sum_{i_1, \dots, i_k} \vec{\alpha}^1_{i_1} \vec{\alpha}^2_{i_2} \cdots \vec{\alpha}^k_{i_k} \, \theta^{i_1} \wedge \theta^{i_2} \wedge \cdots \wedge \theta^{i_k}$$

$$= \sum_{i_1, \dots, i_k} \vec{\alpha}^1_{i_1} \cdots \vec{\alpha}^k_{i_k} \, \epsilon_{i_1 \cdots i_k} \, \theta^1 \wedge \cdots \wedge \theta^k$$

$$= \underline{\det[\vec{\alpha}^1 | \vec{\alpha}^2 | \cdots | \vec{\alpha}^k] \, \theta^1 \wedge \cdots \wedge \theta^k} \; *$$

If $\alpha^1 \wedge \cdots \wedge \alpha^k = 0$ then by $*$ we find $\det[\vec{\alpha}^1 | \cdots | \vec{\alpha}^k] = 0$. However, we also know

$$\vec{\alpha}^1 \wedge \cdots \wedge \vec{\alpha}^k = \underbrace{\det[\vec{\alpha}^1 | \cdots | \vec{\alpha}^k]}_{0} \, e_1 \wedge \cdots \wedge e_k = 0$$

Thus $\underline{\vec{\alpha}^1 \wedge \cdots \wedge \vec{\alpha}^k = 0}$.

P70 Problem 14-5, p. 373 Cartan's Lemma

Let M be smooth n-manifold and let $(\omega^1, .., \omega^k)$ be an ordered k-tuple of smooth one-forms on open $U \subseteq M$ such that $(\omega^1|_p, .., \omega^k|_p)$ is LI for $p \in U$. Given smooth 1-forms $\alpha^1, .., \alpha^k$ on U such that $\sum_{i=1}^{k} \alpha^i \wedge \omega^i = 0$, show that each α^i can be written as linear comb. of $\omega^1, .., \omega^k$ with smooth coefficients.

Since $\omega^1, ..., \omega^k$ LI at $p \in U$ we can extend to basis of one-forms at p, say $\omega^1, .., \omega^k, \omega^{k+1}, .., \omega^n$. Thus $\exists A^i_j, B^i_j \in \mathbb{R}$ such that any one-form at p has

$$\alpha^i = \sum_{j=1}^{k} A^i_j \omega^j + \sum_{j=k+1}^{n} B^i_j \omega^j$$

Assume $\alpha^1, .., \alpha^k$ are one-forms on U with $\sum_{i=1}^{k} \alpha^i \wedge \omega^i = 0$. Consider,

$$0 = \sum_{i=1}^{k} \alpha^i \wedge \omega^i = \sum_{i,j=1}^{k} A^i_j \omega^i \wedge \omega^j + \sum_{i=1}^{k} \sum_{j=k+1}^{n} B^i_j \underbrace{\omega^i \wedge \omega^j}_{\neq 0}$$

Thus $B^i_j = 0$ by LI of $\omega^1, .., \omega^k$ since $i \neq j$.

We're left to analyze the remaining terms with A^i_j, however, we don't need to, $\alpha^i = \sum A^i_j \omega^j$ and smoothness of coefficients follows from smoothness of α^i and ω^j on $U \subseteq M$.

<u>Remark</u>: Mission 5, Advanced Calculus 2015, ~~P109~~ (2013) has more. (this is also a problem in Renteln on p. 50)

$\boxed{P71}$ $\omega = x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy$ $\boxed{\text{Prob. 14-6, p. 375}}$

Let $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$, $z = \rho\cos\phi$

$$dx = \cos\theta\sin\phi\,d\rho - \rho\sin\theta\sin\phi\,d\theta + \rho\cos\theta\cos\phi\,d\phi$$
$$dy = \sin\theta\sin\phi\,d\rho + \rho\cos\theta\sin\phi\,d\theta + \rho\sin\theta\cos\phi\,d\phi$$
$$dz = \cos\phi\,d\rho - \rho\sin\phi\,d\phi$$

Then calculate

① $dy\wedge dz = -\rho\sin\theta\sin^2\phi\,d\rho\wedge d\phi + \rho\sin\theta\cos^2\phi\,d\phi\wedge d\rho$
$+\rho\cos\theta\sin\phi\cos\phi\,d\theta\wedge d\rho - \rho^2\cos\theta\sin^2\phi\,d\theta\wedge d\phi$

$= \rho\sin\theta\,d\phi\wedge d\rho + \rho\cos\theta\sin\phi\cos\phi\,d\theta\wedge d\rho + \rho^2\cos\theta\sin^2\phi\,d\phi\wedge d\theta$

② $dz\wedge dx = -\rho\cos\phi\sin\phi\sin\theta\,d\rho\wedge d\theta + \rho\cos^2\phi\cos\theta\,d\rho\wedge d\phi$
$-\rho\sin\theta\sin^2\phi\,d\phi\wedge d\rho + \rho^2\sin^2\phi\sin\theta\,d\phi\wedge d\theta$

$= \rho\cos\phi\sin\phi\sin\theta\,d\theta\wedge d\rho + \rho\cos\theta\,d\rho\wedge d\phi + \rho^2\sin^2\phi\sin\theta\,d\phi\wedge d\theta$

Woo hoo.

③ $dx\wedge dy = \rho\cos^2\theta\sin^2\phi\,d\rho\wedge d\theta + \rho\cos\theta\cancel{\sin\theta\cos\phi\sin\phi\,d\rho\wedge d\phi}$
$-\rho\sin^2\theta\sin^2\phi\,d\theta\wedge d\rho - \rho^2\sin^2\theta\sin\phi\cos\phi\,d\theta\wedge d\phi$
$+\rho\cos\theta\cancel{\sin\theta\cos\phi\sin\phi}\,d\phi\wedge d\rho + \rho^2\cos^2\theta\sin\phi\cos\phi\,d\phi\wedge d\theta$

$= \rho\sin^2\phi\,d\rho\wedge d\theta + \rho^2\sin\phi\cos\phi\,d\phi\wedge d\theta$ (funny)

Therefore,

$$\omega = (\rho\sin\theta\cdot\rho\cos\theta\sin\phi - \rho\sin\theta\sin\phi\cdot\rho\cos\theta)\,d\phi\wedge d\rho$$
$$+(\rho\cos\theta\sin\phi\cos\phi\cdot\rho\cos\theta\sin\phi + \rho\cos\phi\sin\phi\sin\theta\cdot\rho\sin\theta\sin\phi = \rho^2\sin^2\phi\cos\phi)\,d\theta\wedge d\rho$$
$$+(\rho^3\cos^2\theta\sin^3\phi + \rho^3\sin^3\phi\sin^2\theta + \rho^3\sin\phi\cos^2\phi)\,d\phi\wedge d\theta$$

$$\Rightarrow \omega = \rho^3((\cos^2\theta + \sin^2\theta)\sin^3\phi + \sin\phi\cos^2\phi)\,d\phi\wedge d\theta$$

$$\therefore \boxed{\omega = \rho^3\sin\phi\,d\phi\wedge d\theta}$$

P71 continued

(b.) calculate $d\omega$ in both Cartesian & Sphericals, compare

$$\omega = x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy = \rho^3 \sin\phi \, d\phi\wedge d\theta$$

Hence,

$$d\omega = dx\wedge dy\wedge dz + dy\wedge dz\wedge dx + dz\wedge dx\wedge dy$$

$$\underline{d\omega = 3\,dx\wedge dy\wedge dz = 3\rho^2 \sin\phi \, d\rho\wedge d\phi\wedge d\theta}.$$

(Recall $\dfrac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \rho^2 \sin\phi$ and

notice $dx\wedge dy\wedge dz = \dfrac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}\, d\rho\wedge d\phi\wedge d\theta = \rho^2 \sin\phi\, d\rho\wedge d\phi\wedge d\theta$)

Likewise,

$$d\omega = 3\rho^2 d\rho \wedge \sin\phi\, d\phi\wedge d\theta + \rho^3 \cos\phi \, \overset{0}{\cancel{d\phi\wedge d\phi}}\wedge d\theta$$

$$\underline{d\omega = 3\rho^2 \sin\phi \, d\rho\wedge d\phi\wedge d\theta}.$$

(c.) compute pullback $(\iota_{S^2})^*\omega$ to S^2 using (φ,θ) coord. for sphere S^2 on open subset where φ,θ are defined.
S^2 given by $\rho = 1$ so

$$\boxed{(\iota_{S^2})^*\omega = \sin\phi\, d\phi\wedge d\theta}$$

(d.) $(\iota_{S^2})^*\omega = \sin\phi\, d\phi\wedge d\theta \neq 0$ for $\sin\phi \neq 0$
we cover all of S^2 modulo $(0,0,\pm 1)$ and the angle-jump for θ, let's say $(0,\infty)\times\{0\}\times\mathbb{R}$
But, ω symmetric w.r.t. x,y,z so we can define modified sphericals modulo $(0,\pm 1, 0)$ and half-plane or $(\pm 1, 0, 0)$ and half-plane thus we cover all of S^2 and find $(\iota_{S^2})^*\omega \neq 0$ (hence it's an orientation !)

P71 comment

$$S^2 = F^{-1}\{1\}$$

$$F(x,y,z) = x^2 + y^2 + z^2$$

$$dF = 2x\,dx + 2y\,dy + 2z\,dz$$

$$*(dF) = 2x\,dy\wedge dz + 2y\,dz\wedge dx + 2z\,dx\wedge dy$$

$$\Rightarrow \omega = \frac{1}{2}*(dF) = \frac{1}{2}\Phi_{\nabla F} \leftarrow [\text{flux-form for } \mathbb{R}^3$$

$\uparrow$ Hodge Dual $\qquad \frac{\nabla F}{2} = \langle x, y, z \rangle$

$$d\omega = \frac{1}{2}d\,\Phi_{\nabla F} = \frac{1}{2}(\nabla \cdot \nabla F)\,dx\wedge dy\wedge dz$$

$$d\omega = 3\,dx\wedge dy\wedge dz$$

P72 Problem 14-7, p. 375

In each case, calculate $d\omega$, $F^*\omega$ and verify $F^*(d\omega) = d(F^*\omega)$ for $F: M \longrightarrow N$ given,

(a.) $M = N = \mathbb{R}^2$
$F(s,t) = (st, e^t)$ and $\omega = x\,dy$

$$d\omega = dx \wedge dy$$
$$\begin{aligned} F^* d\omega &= d(F^*x) \wedge d(F^*y) \\ &= d(st) \wedge d(e^t) \\ &= (t\,ds + s\,dt) \wedge e^t dt \\ &= \underline{te^t\, ds \wedge dt}. \end{aligned}$$

$$F^*\omega = (F^*x) F^*dy = st\, d(e^t) = \underline{ste^t dt}.$$

$$\begin{aligned} d(F^*\omega) &= d(ste^t) \wedge dt \\ &= (te^t ds + s(1+t)e^t dt) \wedge dt \\ &= \underline{te^t ds \wedge dt}. \quad \therefore \ \underline{F^* d\omega = d(F^*\omega)}. \end{aligned}$$

(b.) $M = \mathbb{R}^2$, $N = \mathbb{R}^3$, $\omega = y\, dz \wedge dx$
$F(\theta, \varphi) = ((\cos\varphi + 2)\cos\theta, (\cos\varphi + 2)\sin\theta, \sin\varphi)$

$$d\omega = dy \wedge dz \wedge dx = dx \wedge dy \wedge dz$$
$$\begin{aligned} F^* d\omega &= d(\cos\varphi + 2)\cos\theta \wedge d(\cos\varphi + 2)\sin\theta \wedge d\sin\varphi \\ &= (-\sin\varphi\cos\theta\, d\varphi + \gamma(-\sin\theta\, d\theta)) \wedge (-\sin\phi \sin\theta\, d\phi + \gamma\cos\theta\, d\theta) \\ &\qquad \wedge (\cos\varphi\, d\varphi) \\ &= 0 \quad \text{since } \underline{\text{all}} \text{ three forms on } \mathbb{R}^2 \text{ are zero!} \end{aligned}$$

$$\begin{aligned} F^*\omega &= (\cos\varphi + 2)\sin\theta\, (\cos\varphi\, d\varphi) \wedge (-\sin\varphi\cos\theta\, d\varphi + (\cos\varphi + 2)(-\sin\theta\, d\theta)) \\ &= (\cos\varphi + 2)^2 \sin^2\theta \cos\varphi\, d\theta \wedge d\varphi \end{aligned}$$
$$\Rightarrow d(F^*\omega) = 0 \quad \therefore \ \underline{F^* d\omega = d(F^*\omega)}.$$

P72 continued

(c.) $M = \{(u,v) \in \mathbb{R}^2 \mid u^2+v^2<1\}$

$N = \mathbb{R}^3 - \{0\}$, $F(u,v) = (u, v, \sqrt{1-u^2-v^2})$

$$\omega = \frac{1}{(x^2+y^2+z^2)^{3/2}} (x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy)$$

$$\omega = \frac{x}{\rho^3} dy\wedge dz + \frac{y}{\rho^3} dz\wedge dx + \frac{z}{\rho^3} dx\wedge dy$$

$$\omega = \frac{1}{\rho^3}(x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy)$$

$$\omega = \frac{1}{\rho^3}(\rho^3 \sin\phi\, d\phi\wedge d\theta) \quad \text{from P71 b}$$

$$\omega = \sin\phi\, d\phi\wedge d\theta$$

$$d\omega = 0 \hookrightarrow F^*(d\omega) = 0.$$

Likewise, $x=u$, $y=v$, $z=\sqrt{1-u^2-v^2}$ gives $x^2+y^2+z^2=1$

$$F^*\omega = u\,dv\wedge dz + v\,dz\wedge du + z\,du\wedge dv$$

$$= (u\,dv - v\,du)\wedge\left(\frac{1}{2z}(-2u\,du - 2v\,dv)\right) + z\,du\wedge dv$$

$$= \frac{-2u^2}{2z} dv\wedge du + \frac{2v^2}{2z} du\wedge dv + \frac{z^2}{z} du\wedge dv$$

$$= \left(\frac{u^2+v^2+z^2}{z}\right) du\wedge dv$$

$$= \frac{1}{z}(u^2+v^2+1-u^2-v^2)\, du\wedge dv$$

$$= \frac{du\wedge dv}{\sqrt{1-u^2-v^2}}$$

$$d(F^*\omega) = 0 = F^*(d\omega).$$

Remark: ω is the Coulomb field.