Problems are typically taken from either Jeffrey Lee's text *Manifolds and Differential Geometry* (MDG) or John Lee's text *Smooth Manifolds* (SM). I've also written a few problems. Most problems 5pts here.

Problem 73 SM Exercise 19-9 page 495. (local coframe criterion, if you can get part of it good, if all great)

Problem 74 SM Exercise 19-10 page 495. (this problem is ideal)

Problem 75 SM Exercise 19-15 page 499. (flows and integral submanifold verification)

Problem 76 SM Problem 19-2 page 512. (involutivity in view of calculus and algebra of forms)

Problem 77 SM Problem 19-3 page 512. (integrating factor, Pfaff's Theorem)

Problem 78 SM Problem 19-4 page 512. (find a flat chart, I hope this isn't too tricky)

Problem 79 SM Problem 19-5 page 513. (find integral submanifold)

Problem 80 Let G be a Lie group and suppose $\mathfrak{h} \leq \mathfrak{g}$. Explain how the Lie subalgebra $\mathfrak{h}$ can be used to define an involutive distribution on G.

Problem 81 Give five example homogeneous spaces from Chapter 21 of John Lee's text.

Mission 8 Solution

P73 Exercise 19-9, prove Proposition 19.8

D a smooth rank k distribution on n-manifold M and let $\omega^1, \dots, \omega^{n-k}$ be smooth defining forms for D on an open subset $U \subseteq M$. TFAE

(a.) D is involutive on U

(b.) $d\omega^1, \dots, d\omega^{n-k}$ annihilate D (on U)

(c.) $\exists$ smooth 1-forms $\{\alpha^i_j \mid i,j = 1,2,\dots,n-k\}$ such that $d\omega^i = \sum_{j=1}^{n-k} \omega^j \wedge \alpha^i_j$ for $i = 1,\dots,n-k$

Th$^{\underline{m}}$ 19.7 $D \subseteq TM$ a smooth distribution. Then D is involutive iff $\{$If η is any smooth 1-form that annihilates D on an open subset $U \subseteq M$, then $d\eta$ also annihilates D on $U\}$

Assume (a.) If D involutive on U then $\omega^1, \dots, \omega^{n-k}$ are smooth defining forms which annihilate D. Thus Th$^{\underline{m}}$ 19.7 yields $d\omega^1, \dots, d\omega^{n-k}$ also annihilate D. (on U)

$\therefore$ (a) $\Rightarrow$ (b).

Suppose (b.), suppose $d\omega^1, \dots, d\omega^{n-k}$ annihilate D.

Then by Lemma 19.6 taking $\eta = d\omega^i$ we find $\exists$ smooth 1-forms ($P = 2$ so $P - 1 = 1$) $\beta^i_j = \alpha^i_j$ for which $\underline{d\omega^i = \sum_{j=1}^{n-k} \omega^i \wedge \alpha^i_j}$ $*$ thus (b.) $\Rightarrow$ (C.).

<u>Assume (c.)</u> hence $*$ holds true on U for some smooth 1-forms α^i_j. Notice $*$ implies $d\omega^1, \dots, d\omega^{n-k}$ annihilate D on U

(it remains to show D is involutive on $U \subseteq M$)

P74 Exercise 19-10

For any smooth dist. $D \subseteq TM$, show $\mathcal{J}(D)$ is an ideal in $\Omega^* M$

Def^n/ An ideal in $\Omega^* M$ is a linear subspace $\mathcal{J} \subseteq \Omega^*(M)$ that is closed under wedge products of $\Omega^* M$ elements; $\omega \in \mathcal{J} \Rightarrow \eta \wedge \omega \in \mathcal{J} \;\; \forall\, \eta \in \Omega^* M$.

Suppose D has annihilating one-forms $\omega^1, \dots, \omega^{n-k}$. Let $\gamma \in \mathcal{J}(D)$ then γ annihilates D thus, by Lemma 19.6, $\exists$ smooth forms β^i such that $\gamma = \sum_{i=1}^{n-k} \omega^i \wedge \beta^i$. Let $\eta \in \Omega^*(M)$ and consider $\eta \wedge \gamma$. We find

$$\begin{aligned}
\eta \wedge \gamma &= \eta \wedge \left(\sum_{i=1}^{n-k} \omega^i \wedge \beta^i \right) \\
&= \sum_{i=1}^{n-k} \eta \wedge \omega^i \wedge \beta^i \\
&= \sum_{i=1}^{n-k} \omega^i \wedge \Big(\underbrace{(-1)^{\deg(\eta)} \eta \wedge \beta^i}_{\text{smooth form}} \Big)
\end{aligned}$$

Thus $\eta \wedge \gamma \in \mathcal{J}(D)$ by Lemma 19.6 once more. Thus $\mathcal{J}(D)$ is an ideal in $\Omega^* M$. //

$\boxed{P\,75}$ $V = \partial_x + y\partial_z \qquad \alpha_t(x,y,z) = (x+t,\ y,\ z+ty)$

$W = \partial_y + x\,\partial_z \qquad \beta_t(x,y,z) = (x,\ y+t,\ z+tx)$

Have flows as above and also that level sets $z - xy$ are integral manifolds of D.

(Ex 19-15, p. 499 of Lee)

$V: \quad \dfrac{dx}{dt} = 1$ and $\dfrac{dy}{dt} = 0$ and $\dfrac{dz}{dt} = y$ (DEq^{ns} for $\int$-curve of V)

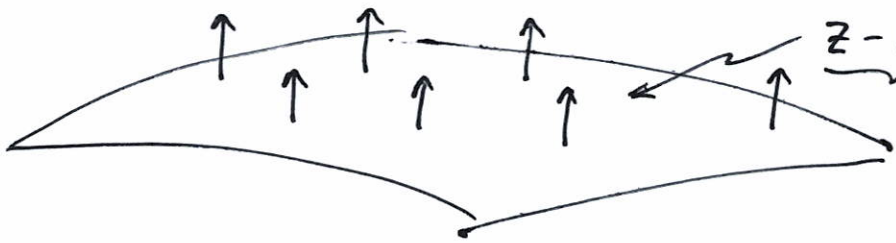
$x = x_0 + t \qquad y = y_0 \qquad z = y_0 t + z_0$

$$\therefore\ \underline{\alpha_t(x,y,z) = (x+t,\ y,\ z+yt)}.$$

$W: \quad \dfrac{dx}{dt} = 0$ and $\dfrac{dy}{dt} = 1$ and $\dfrac{dz}{dt} = x$ (DEq^{ns} for $\int$-curve of W)

$x = x_0 \qquad y = t + y_0 \qquad z = x_0 t + z_0$

$$\underline{\beta_t(x,y,z) = (x,\ y+t,\ z+tx)}.$$



$\underbrace{z - xy}_{F} = C$

$\nabla F = \langle -y, -x, 1\rangle$

$$\langle V, \nabla F\rangle = \langle\langle 1,0,y\rangle, \langle -y,-x,1\rangle\rangle = -y + y = 0$$

$$\langle W, \nabla F\rangle = \langle\langle 0,1,x\rangle, \langle -y,-x,1\rangle\rangle = -x + x = 0$$

Thus $V, W \perp \nabla F$

$\Rightarrow$ V, W are tangent to $z - xy = C$.

$\therefore$ $z - xy = C$ are integral manifolds of D. //

P76 Problem 19-2

Let D be smooth, rank k distribution on smooth n-manifold M, and $\omega^1, \omega^2, \dots, \omega^{n-k}$ are smooth local defining forms for D on $U \subseteq M$. Show D is involutive on U iff the following identity holds for each $i = 1, \dots, n-k$,

$$d\omega^i \wedge \omega^1 \wedge \cdots \wedge \omega^{n-k} = 0$$

$\Rightarrow$ If D is involutive then as $\omega^1, \dots, \omega^{n-k}$ annihilate D on U we have also $d\omega^i$ annihilate D on U by Prop. 19.8 b. But then $d\omega^i = \sum_{j=1}^{n-k} \omega^j \wedge \alpha^i_j$ by Prop. 19.8c

and so

$$d\omega^i \wedge \omega^1 \wedge \cdots \wedge \omega^{n-k} = \left(\sum_{j=1}^{n-k} \omega^j \wedge \alpha^i_j \right) \wedge \underbrace{\omega^1 \wedge \cdots \wedge \omega^{n-k}}_{*}$$

$$= 0$$

since $\omega^j \wedge \omega^j = 0$ and every term in the sum has duplicate ω^j in $*$.

$\Leftarrow$ Assume that

$$d\omega^i \wedge \omega^1 \wedge \cdots \wedge \omega^{n-k} = 0 \quad \star$$

Since $\omega^1, \dots, \omega^{n-k}$ are LI at a pt, we find $d\omega^i$ must linearly depend on $\omega^1, \dots, \omega^{n-k}$ since $d\omega^i, \omega^1, \dots, \omega^{n-k}$ is linearly dependent by the given eq$^{\underline{n}}$ $\star$. Thus $\exists\, \alpha^i_j$ smooth one-forms for which $d\omega^i = \sum \omega^j \wedge \alpha^i_j$ for $i = 1, 2, \dots, n-k$. Thus D is involutive by Prop. 19.8c. //

P77

PROBLEM 19-3

Let ω be smooth 1-form on M. A smooth positive function μ on some open subset $U \subseteq M$ is called an <u>integrating factor</u> for ω if $\mu\omega$ is exact on U.

(a.) If $\omega \neq 0$ on M, then ω admits $\int$-factor in nbhd of each pt iff $d\omega \wedge \omega \equiv 0$

(b.) If $\dim M = 2$, then every nonvanishing smooth 1-form an integrating factor in nbhd of each pt.

(a.) $\Rightarrow$] If $\exists \mu$ s.t. $\mu\omega = d\alpha$ on U then notice $d\mu \wedge \omega + \mu d\omega = d(d\alpha) = 0$

thus $d\omega = \frac{-1}{\mu} d\mu \wedge \omega$ and $d\omega \wedge \omega = 0$ on U

Thus $d\omega \wedge \omega \equiv 0$ since U was arbitrary.

$\Leftarrow$] Suppose $d\omega \wedge \omega \equiv 0$. (see Prop 19.8 b I think)

Then it follows that the distribution with local annihilating form ω is involutive on M.

Hence, by Frobenius Th^m, this $(n-1)$-dimensional distribution on TM is completely integrable. Thus $\exists (n-1)$-dim'l integrable manifolds, aligned with D.

At a given local, we can write the $\int$-manifold U as level surface $F(x) = 0$ and $dF = 0$ on U

(Lemma 19.6) $\begin{pmatrix} \eta = dF \\ \beta = \mu \end{pmatrix}$ Then $dF = \omega \wedge \mu$ for 0-form μ and $\mu > 0$ can be imposed by replacing F with $-F$ if needed. //

(b.) For $\dim M = 2$, $d\omega \wedge \omega = 0$ since $\Lambda^3 M \equiv 0$

thus, by (a.) there exists integrating factor in some nbhd of each pt. on M.

Example 19.14 : — (calculations here intend to help with P78, currently unsolved, Problem 19-4 in Lee) —

D spanned by X and Y

$$X = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + x(y+1)\frac{\partial}{\partial z}$$

$$Y = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$$

we can calculate $[X, Y] = -Y$.

- GOAL, find flat chart at origin
  - $X\big|_0 = \frac{\partial}{\partial y}$ and $Y\big|_0 = \frac{\partial}{\partial x}$ } complementary to $\frac{\partial}{\partial z}$

$\pi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ by $\pi(x, y, z) = (x, y)$

$d\pi\big|_{D(x,y,z)} : D(x,y,z) \longrightarrow T_{(x,y)}\mathbb{R}^2$

- if we can find smooth local sections V, W that are π-related to $\partial/\partial x$ and $\partial/\partial y$ respectively

so... V, W must be commuting

$$\cancel{d\pi\,\partial(X)}\ d\pi(V) = \frac{\partial}{\partial x}$$

$$d\pi(W) = \frac{\partial}{\partial y}$$

$$d\pi_p(V) = \frac{\partial}{\partial x}\bigg|_{\pi(p)}$$

$$d\pi_p(W) = \frac{\partial}{\partial y}\bigg|_{\pi(p)}$$

p. 182 F-related

$X(f \circ F) = (Yf) \circ F$

$X(f \circ F)(p) = dF_p(X_p)(f)$

$dF_p(X_p) = Y_{F(p)}$

X & Y are F-related

$$\left.\begin{aligned} d\pi(\partial_x) &= \partial_x \\ d\pi(\partial_y) &= \partial_y \\ d\pi(\partial_z) &= 0 \end{aligned}\right\}$$

all from $\pi(x, y, z) = (x, y)$ so we have freedom to put whatever next to ∂_z

$$V = \frac{\partial}{\partial x} + a\frac{\partial}{\partial z}$$

$$W = \frac{\partial}{\partial y} + b\frac{\partial}{\partial z}$$

$$[V, W] = [\partial_x + a\partial_z,\ \partial_y + b\partial_z]$$

$$= ([V,W]x)\partial_x + ([V,W]y)\partial_y + ([V,W]z)\partial_z$$

$$[V,W]f = (\partial_x + a\partial_z)(\partial_y f + b\partial_z f) - (\partial_y + b\partial_z)(\partial_x f + a\partial_z f)$$

$$= \cancel{\partial_x\partial_y f} + \underline{(\partial_x b)(\partial_z f)} + \cancel{b\partial_x\partial_z f}$$

$$+ a\underline{\partial_z\partial_y f} + \underline{a(\partial_z b)(\partial_z f)} + \underline{ab\partial_z^2 f}$$

$$- \cancel{\partial_y\partial_x f} - \underline{(\partial_y a)(\partial_z f)} - a\underline{\partial_y\partial_z f} - \cancel{b\partial_z\partial_x f}$$

$$- \underline{b(\partial_z a)(\partial_z f)} - \underline{ba\partial_z^2 f}$$

$$= (\underbrace{\partial_x b + a\partial_z b - \partial_y a - b\partial_z a}_{\text{needs to be zero for } [V,W]=0})\partial_z f$$

needs to be zero for $[V, W] = 0$

Lee suggests setting $a = u = y$
$b = v = x$

$$V = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$$ find flow from $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = 0$, $\frac{dz}{dt} = y$

$$\begin{cases} x = x_0 + t \\ y = y_0 \\ z = y_0 t + z_0 \end{cases}$$

$$\underline{\alpha_t(x, y, z) = (x + t,\ y,\ yt + z)}.$$

$$W = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$$

$\frac{dx}{dt} = 0$
$\frac{dy}{dt} = 1$
$\frac{dz}{dt} = x$

$$\underline{\beta_t(x, y, z) = (x,\ y + t,\ tx + z)}.$$

following procedure of Example 9.47

$$\Phi(u,v,w) = (\alpha_u \circ \beta_v)(0,0,w)$$
$$= \alpha_u(\beta_v(0,0,w))$$
$$= \alpha_u(0,v,w)$$
$$= (u, v, w+uv)$$

using flow formulas from last pg

$$\Phi(u,v,w) = (x,y,z) = (u,v,w+uv)$$

$x = u$

$y = v$

$z = w + xy$

$w = z - xy$

$$\Phi^{-1}(x,y,z) = (x, y, z-xy)$$

Show level set of $z-xy$ is $\int$-manifold of D.

$$F(x,y,z) = z - xy = c$$

$$\omega = dz - y\,dx - x\,dy$$

$$\omega(X) = x(y+1) - yx - x = 0$$

$$\omega(Y) = -y + y = 0$$

$$\therefore \quad \omega(fX + gY) = 0$$

$$\Rightarrow \quad \omega|_D = 0 \quad \therefore$$

D is integral to $F = c$.

$$\mathbb{X} = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \qquad \mathbb{Y} = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$$

$$\frac{dx}{dt} = 0 \qquad \frac{dz}{dt} = y \qquad \frac{dy}{dt} = -z$$

$$\beta(t) = (x_0 + z_0 \sin t,\ y_0,\ z_0 \cos t)$$

$$-y'' = y$$

$$y = y_0 \cos(t)$$

$$z = +y_0 \sin(t) + z_0$$

$$\gamma_{\mathbb{X}}(t) = (x_0,\ y_0 \cos t,\ z_0 + y_0 \sin t)$$

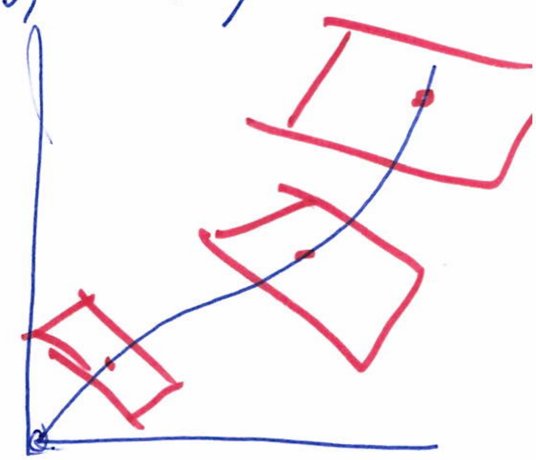
$$\gamma_{\mathbb{X}}(0) = (x_0, y_0, z_0)$$

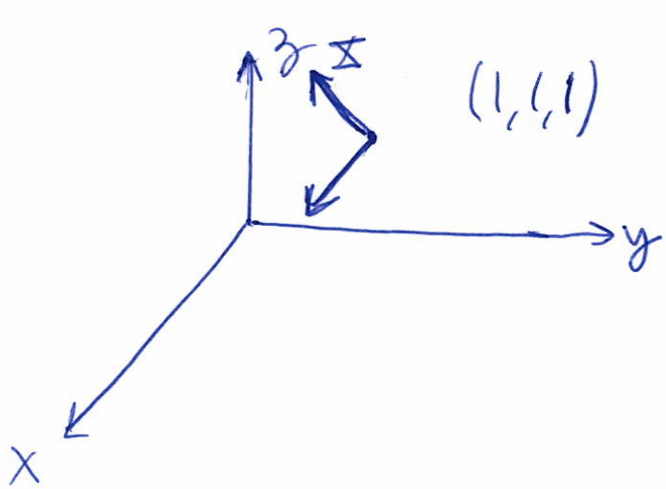
$$\alpha_u(x,y,z) = (x,\ y\cos u,\ z + y \sin u)$$

$$\Phi(v,w,w) \qquad \beta_v(x,y,z) = (x + z\sin v,\ y,\ z\cos v)$$

$$\cancel{\Phi_{u,w}}$$

$$\Phi(u,v,w) = (\alpha_u \circ \beta_v)(w,w,w)$$





$$① \quad \omega(X) = 0 \qquad \omega = a\,dx + b\,dy + c\,dz$$

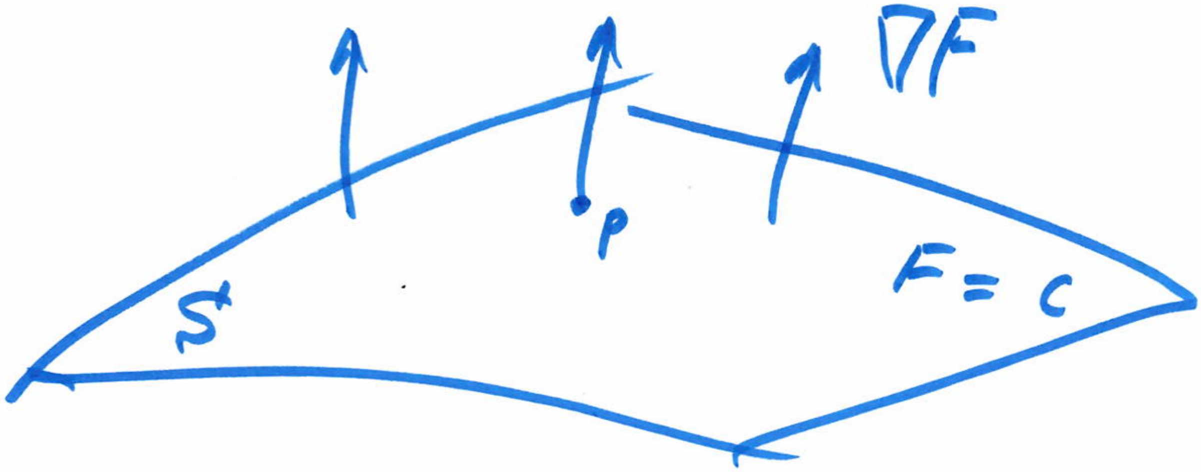
$$② \quad \omega(Y) = 0$$

$$① \quad yc - zb = 0 \qquad yc = bz \hookrightarrow \underline{c = \frac{z}{y} b}.$$

$$② \quad az - cx = 0 \qquad cx = az \hookrightarrow a = \frac{x}{z} c$$

$$a = \frac{x}{z}\frac{z}{y} b = \frac{x}{y} b$$

$$\begin{aligned}
\omega &= a\,dx + b\,dy + c\,dz \\
&= \left(\frac{x}{y}\right) b\,dx + b\,dy + \frac{z}{y} b\,dz \\
&= \frac{b}{}\left(\frac{x}{y}\,dx + dy + \frac{z}{y}\,dz\right) \\
&= \frac{b}{y}\left(x\,dx + y\,dy + z\,dz\right) \\
&= \frac{b}{2y}\,d\left(x^2 + y^2 + z^2\right)
\end{aligned}$$



$$T_p S = (\nabla F)(p)^{\perp}$$

$$= \mathrm{ann}(\omega_p)$$

$$= \mathrm{span}\{ \mathbf{X}_u, \mathbf{X}_v \} \sim \text{at } p$$

$$p = \mathbf{X}(u_0, v_0)$$

$$S = \{ (x,y,z) \mid F(x,y,z) = c \} \quad \text{level surface}$$

$$= \{ \mathbf{X}(u,v) \mid (u,v) \in D \} \quad \text{parametric}$$

$D \sim$ two-dim'l dist.

$$\mathbf{X}_u, \mathbf{X}_v \in D_p \quad \forall (u,v) \in D$$

$$p = \mathbf{X}(u,v)$$

$$D_p = \mathrm{ann}(\omega_p)$$

$\boxed{P79}$ Problem 19-5

Let D be the distribution on $\mathbb{R}^3$ spanned by

$$X = \frac{\partial}{\partial x} + yz\frac{\partial}{\partial z} \qquad Y = \frac{\partial}{\partial y}$$

(a.) find an integral submanifold of D passing through $(0,0,0)$.

(b.) Is D involutive? Explain answer in light of part (a.)

(a.) Consider $S = \{(x,y,0) \mid x,y \in \mathbb{R}\}$ (the xy-plane)

then $X|_S = \frac{\partial}{\partial x}$ and $Y|_S = \frac{\partial}{\partial y}$ which are clearly tangent to T_pS which has basis $\partial_x|_p, \partial_y|_p$

In short S is an integral submanifold of D passing through $(0,0,0)$.

(b.) We can calculate

$$[X, Y] = -z\frac{\partial}{\partial z} \notin D$$

thus D is not involutive on $\mathbb{R}^3$. However, if we restrict D to S then $D|_S$ is involutive and in this rather silly case S itself is the integrable submanifold (I suppose any subset of S would also do)

<u>Remark</u>: existence of an integral submanifold to D at some point does not imply the existence of integral submanifolds everywhere else. This problem illustrates this phenomenon.

$\boxed{P80}$ Let G be a Lie group and suppose $\mathfrak{h} \leq \mathfrak{g}$. Explain how the Lie subalgebra $\mathfrak{h}$ can be used to define an involutive distribution on G

$$T_eG \cong \mathfrak{g} \cong LIVF(G)$$

We can construct $\mathcal{H} \subseteq LIVF(G)$ by pushing forward $\mathfrak{h} \leq T_eG = \mathfrak{g}$. Ok, so, $\exists$ subset of T_eG which is isomorphic to $\mathfrak{h}$ as a Lie subalgebra and for $V_0 \in \mathfrak{h} \subseteq T_eG$ define $l(V_0) \in \mathfrak{X}(G)$ by

$$l(V_0)(g) = d_eL_g(V_0) \in T_g G$$

But then we could prove if $l(V_0), l(W_0) \in \mathcal{H}$ then,

$$[l(V_0), l(W_0)] = l[V_0, W_0]$$

and as $V_0, W_0 \in \mathfrak{h} \Rightarrow [V_0, W_0] \in \mathfrak{h}$ we find $[l(V_0), l(W_0)] \in \mathcal{H}$. Thus $\mathcal{H}$ is involutive.

$\boxed{P81}$ See p. 553 $O(n) = \{A \in \mathbb{R}^{n\times n} \mid A^TA = I\}$

$$S^{n-1} \approx O(n)/O(n-1) \qquad \mathbb{R}^n \approx E(n)/O(n) \qquad S^{2n-1} \approx \frac{U(n)}{U(n-1)}$$

$$n \geq 2, \quad S^{n-1} \approx SO(n)/SO(n-1) \qquad U \approx \frac{SL(2,\mathbb{R})}{SO(2)}$$