Problems are typically taken from either Jeffrey Lee's text *Manifolds and Differential Geometry* (MDG) or John Lee's text *Smooth Manifolds* (SM). I've also written a few problems. Most problems 5pts here.

Problem 82 Suppose $\{e_1, e_2, e_3\}$ is a frame at $p = (1, 2, 3)$ and $\{g_1, g_2, g_3\}$ is a frame at $q = (1, 1, 1)$. Furthermore, you're given

$$e_1 = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle, \quad e_1 = \left\langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0 \right\rangle, \quad e_3 = \langle 0, 0, 1 \rangle$$

and $g_1 = \langle 0,\ 1,\ 0 \rangle$, $g_2 = \langle 0,\ 0,\ 1 \rangle$, $g_3 = \langle 1,\ 0,\ 0 \rangle$. Find an isometry F which maps p to q and has a tangent map F_* for which $F_*(e_i) = g_i$ for $i = 1, 2, 3$.

Problem 83 Prove the Frenet Serret Equations for a nonstop, non-linear path in $s \mapsto \gamma(s) \in \mathbb{R}^3$:

$$\frac{dT}{dt} = \kappa v N, \qquad \frac{dN}{dt} = -\kappa v T + \tau v B, \qquad \frac{dB}{dt} = -\tau v N.$$

Recall we define $T = \frac{1}{v}\gamma'$ where $v = \|\gamma'\|$ and $N = \frac{1}{\|T'\|}T'$ and $B = T \times N$. You will also need to know the definitions of curvature $\kappa = \frac{1}{v}\|T'\|$ and torsion $\tau = -\frac{1}{v}N \bullet B'$.

• **Problem 84** Let α be a unit-speed, non-linear curve. Then α is a planar curve if and only if $\tau = 0$.

• **Problem 85** Generally two parametrized curves $\alpha : I \to \mathbb{R}^3$ and $\beta : I \to \mathbb{R}^3$ are **congruent** if there exists an isometry F for which $\beta = F \circ \alpha$. This problem asks for you to show if two unit-speed curves are congruent then they have the same curvature and (up to a sign) torsion. The converse is also true, in fact, arclength parametrized curves are congruent if and only if they have same curvature function and upto a sign the same torsion[1].

(a.) Let α be a nonlinear arclength parametrized curve with Frenet frame T, N, B and curvature κ and torsion τ. Suppose $F : \mathbb{R}^3 \to \mathbb{R}^3$ is an Euclidean isometry. Show that $\bar{\alpha} = F \circ \alpha$ has Frenet frame $\bar{T}, \bar{N}, \bar{B}$ with curvature $\bar{\kappa}$ and torsion $\bar{\tau}$ where

$$\bar{T} = F_*(T), \quad \bar{N} = F_*(N), \quad \bar{B} = \det(F_*)F_*(B), \quad \bar{\kappa} = \kappa, \quad \bar{\tau} = det(F_*)\tau$$

(b.) Let α be an arclength parametrized curve. Then α has constant curvature and torsion if and only if α is a helix.

Problem 86 Let A_{ik} be a p-form and B_{kj} be a q-form for $1 \le i \le m$, $1 \le k \le r$ and $1 \le j \le n$ then we say A is an $m \times r$ matrix of p-forms and B is a $r \times n$ matrix of q-forms. Then we say A and B are **multipliable** and define the $m \times n$-matrix of $(p+q)$-forms $A \wedge B$ by:

$$(A \wedge B)_{ij} = \sum_{k=1}^{r} A_{ik} \wedge B_{kj}.$$

[1] This is my Theorem 3.3.11 on page 72 of my Differential Geometry notes from 2021. You might need to assume the converse to argue part (b.).

Likewise, we denote the **exterior derivative of** A by dA which is defined to be the $m \times r$-matrix of $(p+1)$-forms given by $(dA)_{ik} = dA_{ik}$. Let A be a matrix of p-forms and$B$ be a matrix of q-forms and suppose A, B are multipliable then show

$$d(A \wedge B) = dA \wedge B + (-1)^p A \wedge dB.$$

Also, for a matrix of functions A for which $A^T A = I$ show $A \wedge dA^T \wedge A = -dA$

Problem 87 Suppose $W \in \mathfrak{X}(\mathbb{R}^3)$ and $v \in T_p\mathbb{R}^3$. The **covariant derivative** of W with respect to v at p is the tangent vector:

$$(\nabla_v W)(p) = W(p+tv)'(0) \in T_p\mathbb{R}^3.$$

If $V \in \mathfrak{X}(\mathbb{R}^3)$ and $V(p) = v_p$ then the assignment $p \to (\nabla_{v_p} W)(p)$ defines $\nabla_V W \in \mathfrak{X}(\mathbb{R}^3)$ and we say $\nabla_V W$ is the **covariant derivative** of W with respect to V. This derivative measures how W changes in the V-direction. A nice coordinate formula for the covariant derivative is simply:[2]

$$\nabla_V W = \sum_{j=1}^{3} V[W^j] U_j$$

where $U_i = \frac{\partial}{\partial x^i}$ for $i = 1, 2, 3$ and $W = \sum_j W^j U_j$.

(a.) $\nabla_{fV} W = f \nabla_V W$ for all smooth $f : \mathbb{R}^3 \to \mathbb{R}$

(b.) $\nabla_V (fW) = V[f]\, W + f \nabla_V W$ for all smooth $f : \mathbb{R}^3 \to \mathbb{R}$

(c.) $U[V \bullet W] = \nabla_U V \bullet W + V \bullet \nabla_U W$

(d.) If $\|V\| = 1$ then $\nabla_V (V) = 0$

Problem 88 If $\{E_1, E_2, E_3\}$ is an orthonormal frame for $\mathbb{R}^3$ and $V \in \mathfrak{X}(\mathbb{R}^3)$ then there exist functions f^1, f^2, f^3 called the **components of** V with respect to the frame $\{E_1, E_2, E_3\}$

$$V = f^1 E_1 + f^2 E_2 + f^3 E_3$$

Also we define $\omega_{ij}(p) \in (T_p\mathbb{R}^3)^*$ by

$$\omega_{ij}(p)(v) = (\, \nabla_v E_i \,) \bullet E_j(p)$$

for each $v \in T_p\mathbb{R}^3$. That is, ω_{ij} is a differential one-form on $\mathbb{R}^3$ defined by the assignment $p \mapsto \omega_{ij}(p)$ for each $p \in \mathbb{R}^3$ and it is known as a **connection form**. Suppose $V, W \in \mathfrak{X}(\mathbb{R}^3)$ where $V = \sum f^i E_i$ and $W = \sum_j g^j E_j$. Then notice $\sum_j \omega_{ij}(V) E_j = \nabla_V E_i$ hence

$$\nabla_V W = \nabla_V \left(\sum_i g^i E_i \right) = \sum_i (V[g^i] E_i + g^i \nabla_V E_i) = \sum_i \left(V[g^i] E_i + g^i \sum_j \omega_{ij}(V) E_j \right)$$

then relabling the first sum and exchanging the order of the second we derive

$$\nabla_V W = \sum_j \left(V[g^j] + \sum_i g^i \omega_{ij}(V) \right) E_j.$$

In short, finding the formulas for the connection forms allows us to capture the change in vector fields which is due to the change in the orthonormal frame over the space. Sorry for the lengthy preamble, now for the actual problem:

[2] perhaps I'll derive this in lecture, if not, it's on page 45 of my 2021 Differential Geometry notes

(a.) If $A_{ij} = E_i \bullet U_j$ then show $E_i = \sum_j A_{ij} U_j$ and $\theta^i = \sum_j A_{ij} dx^j$. We call A the **attitude matrix** of the frame E_1, E_2, E_3.

Notation: It is useful to use matrix notation to think about a column of one-forms, we can restate the proposition above as follows:

$$\theta = \begin{bmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{bmatrix} \qquad \& \qquad d\xi = \begin{bmatrix} dx^1 \\ dx^2 \\ dx^3 \end{bmatrix} \quad \Rightarrow \quad \theta = A d\xi.$$

Just to be explicit,

$$Ad\xi = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} dx^1 \\ dx^2 \\ dx^3 \end{bmatrix} = \begin{bmatrix} A_{11}dx^2 + A_{12}dx^2 + A_{13}dx^3 \\ A_{21}dx^2 + A_{22}dx^2 + A_{23}dx^3 \\ A_{31}dx^2 + A_{32}dx^2 + A_{33}dx^3 \end{bmatrix} = \begin{bmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{bmatrix} = \theta.$$

(b.) show $\omega = dA \wedge A^T$.

(c.) derive Cartan's Structure Equations for $\mathbb{R}^3$; $d\theta = \omega \wedge \theta$ and $d\omega = \omega \wedge \omega$.
To be explicit,

$$d\theta^i = \sum_j \omega_{ij} \wedge \theta^j \qquad \& \qquad d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj}.$$

Problem 89 Let α be a unit speed curve with $\kappa > 0$ and suppose E_1, E_2, E_3 is an orthonormal frame field on $\mathbb{R}^3$ such that the frame restricts to the Frenet frame T, N, B on α. Show the connection forms of the frame are precisely the curvature and torsion of the curve. Here you probably will find the identity $\nabla_{\alpha'(s)} W = \frac{d}{ds}(W \circ \alpha)(s)$ is helpful.

Problem 90 If $\phi = \sum_i f_i \theta^i$ where θ^i are dual to orthonormal frame E_i with connection forms ω_{ij} then $d\phi = \sum_j [df_j + \sum_i f_i \omega_{ij}] \wedge \theta^j$ (you can and should make use of Cartan's Structure Equations if helpful)

Problem 91 Let U be the unit-normal to a surface M in $\mathbb{R}^3$. Let $p \in M$, if $v \in T_pM$ then define the **shape operator** at p acting on v by $S_p(v) = -(\nabla_v U)(p)$. Show the following:

(a.) $S_p : T_pM \to T_pM$ and S_p is a linear transformation.

(b.) $S_p(v) \bullet w = v \bullet S_p(w)$ or equivalently, the matrix of S_p is symmetric

(c.) $S_p(v) \times S_p(w) = K(p) v \times w$ and $S_p(v) \times w + v \times S_p(w) = 2H(p) v \times w$ where K is the Gaussian curvature (defined by $K(p) = det(S_p)$) and H is the mean curvature (defined by $H(p) = trace(S_p)$)

(d.) If M is a surface with tangent vectors $v_1, v_2 \in T_pM$ such that $S_p(v_1) = 6v_1$ and $S_p(v_2) = 7v_2$ then find $K(p)$ and $H(p)$.

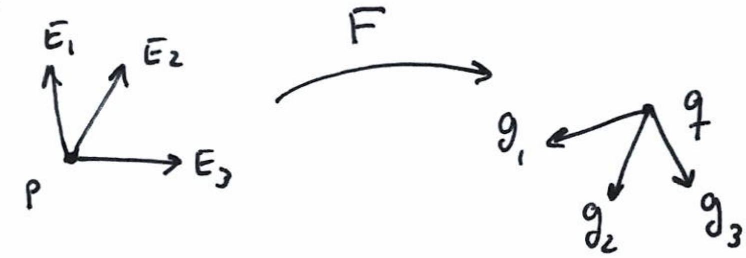
• **Problem 92** Use the E, F, G, L, M, N formulas for Gaussian curvature K and mean curvature H to find K and H for

(a.) the Helicoid $X(u, v) = \langle u \cos v, u \sin v, bv \rangle$ where $b \neq 0$ is a constant

(b.) the Cylinder $X(u, v) = \langle R \cos u, R \sin u, v \rangle$ where $R > 0$ is a constant

SOLUTION TO MISSION 9

P82



$$e_1 = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle$$
$$e_2 = \left\langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0 \right\rangle$$
$$e_3 = \langle 0, 0, 1 \rangle$$
$$g_1 = \langle 0, 1, 0 \rangle$$
$$g_2 = \langle 0, 0, 1 \rangle$$
$$g_3 = \langle 1, 0, 0 \rangle$$
$$p = (1, 2, 3)$$
$$q = (1, 1, 1)$$

$$F(x) = Rx + b \qquad F_* v = Rv$$

$$F_* e_i = g_i \quad \longleftrightarrow \quad \begin{aligned} Re_1 &= g_1 \\ Re_2 &= g_2 \\ Re_3 &= g_3 \end{aligned}$$

$$[Re_1 \mid Re_2 \mid Re_3] = [g_1 \mid g_2 \mid g_3]$$
$$R\,[e_1 \mid e_2 \mid e_3] = [g_1 \mid g_2 \mid g_3]$$
$$RE = G \quad \therefore \quad \underline{R = GE^T}. \qquad (E^T E = I)$$

$$F(x) = GE^T x + b$$
$$F(p) = q \Rightarrow GE^T p + b = q$$
$$\therefore \quad \underline{b = q - GE^T p = q - Rp}.$$

$$R = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}$$

$$b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 - 3 \\ 1 - 3/\sqrt{2} \\ 1 + 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -2 \\ 1 - 3/\sqrt{2} \\ 1 + 1/\sqrt{2} \end{bmatrix}$$

$$\boxed{F(x) = \begin{bmatrix} 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -2 \\ \frac{\sqrt{2} - 3}{\sqrt{2}} \\ \frac{\sqrt{2} + 1}{\sqrt{2}} \end{bmatrix}}$$

$\boxed{P83}$ we solved in Lecture, essentially the argument is this, since T, N, B form a frame (orthonormal in fact) thus

$$① \quad T' = (T' \cdot T)T + (T' \cdot N)N + (T' \cdot B)B$$

$$② \quad N' = (N' \cdot T)T + (N' \cdot N)N + (N' \cdot B)B$$

$$③ \quad B' = (B' \cdot T)T + (B' \cdot N)N + (B' \cdot B)B$$

But $A \cdot A = 1 \Rightarrow A' \cdot A + A \cdot A' = 0 \Rightarrow A \cdot A' = 0$

thus $T \cdot T = 1$, $N \cdot N = 1$, $B \cdot B = 1$ yields

$T' \cdot T = N' \cdot N = B' \cdot B = 0$. Furthermore,

we define $N = \dfrac{T'}{\|T'\|} \therefore T' = \|T'\| N$

and so as $\kappa = \dfrac{1}{v}\|T'\|$ we find $\boxed{T' = \kappa v N}$

Thus $T' \cdot B = 0$. This finishes ①. Now

$$N \cdot T = 0 \Rightarrow N' \cdot T + N \cdot T' = 0$$
$$\Rightarrow N' \cdot T = -N \cdot T' = -N \cdot (\kappa v N) = -\kappa v$$

$$N \cdot B = 0 \Rightarrow N' \cdot B + N \cdot B' = 0$$

$$\Rightarrow N' \cdot B = -N \cdot B' = \tau v$$

Thus, ② reads $\boxed{N' = -\kappa v T + \tau v B}$

$$B \cdot T = 0 \Rightarrow B' \cdot T + B \cdot T' = 0$$
$$\Rightarrow B' \cdot T = -B \cdot T' = -B \cdot (\kappa v N) = 0.$$

Thus, ③ reads $\boxed{B' = -\tau v N}$

P84 Let α be a unit-speed, non-linear curve.
Show α is planar curve $\Leftrightarrow \tau = 0$

$\Rightarrow$ Suppose α is unit-speed, non-linear curve for which $\alpha(s) \in \mathcal{P} = \underbrace{\vec{r}_0 + \langle a,b,c\rangle^{\perp}}_{\text{base point } \vec{r}_0}$ and $\underline{\text{unit}}$ $\underline{\text{normal}}$ $\underbrace{\langle a,b,c\rangle}_{V}$

If $s_0 \in \mathrm{dom}(\alpha)$ then

$$\alpha(s) - \alpha(s_0) \in \mathcal{P} \Rightarrow (\alpha(s) - \alpha(s_0)) \cdot V = 0$$

then differentiate,

$$\alpha'(s) \cdot V = 0 \Rightarrow \underline{T \cdot V = 0}.$$

$$\alpha''(s) \cdot V = 0 \Rightarrow T' \cdot V = \kappa N \cdot V = 0$$

But, $\kappa \neq 0$ as α non-linear, thus $T \cdot V = 0$ and $N \cdot V = 0$ at each $s \in \mathrm{dom}(\alpha) \Rightarrow B(s) = \pm V$

Thus $B' = 0 = -\tau N \therefore \boxed{\tau = 0}$ //

$\Leftarrow$ Suppose $\tau = 0$. Then $\frac{dB}{ds} = -\tau N = 0 \quad \forall s$ thus $B(s) = B(s_0) \quad \forall s \in \mathrm{dom}(\alpha)$ (again assume α is unit-speed, nonlinear curve so the Frenet apparatus is well-defined). Define,

$$f(s) = [\alpha(s) - \alpha(s_0)] \cdot B(s_0)$$

Calculate,

$$f'(s) = \alpha'(s) \cdot B(s_0) = T(s) \cdot B(s) = 0$$

for each $s \in \mathrm{dom}(\alpha)$. Thus f is constant and we've shown $f(s) = f(s_0) = (\alpha(s_0) - \alpha(s_0)) \cdot B(s_0) = 0$

$$\therefore (\alpha(s) - \alpha(s_0)) \cdot B(s_0) = 0$$

Hence α is planar curve on plane with normal $B(s_0)$. //

$\boxed{P85}$ (a.) Let α be nonlinear, arclength-parametrized curve with Frenet frame T, N, B and curvature κ and torsion τ. Let $F: \mathbb{R}^3 \to \mathbb{R}^3$ be a Euclidean isometry. Show $\bar{\alpha} = F \circ \alpha$ has Frenet frame $\bar{T}, \bar{N}, \bar{B}$ with curvature $\bar{\kappa}$ and torsion $\bar{\tau}$ s.t.

$$\bar{T} = F_*(T), \quad \bar{N} = F_*(N), \quad \bar{B} = \det(F_*) F_*(B)$$

and $\bar{\kappa} = \kappa$ and $\bar{\tau} = \det(F_*)\tau$

$F \in \mathrm{Isom}(\mathbb{R}^3) \Rightarrow F(x) = Rx + b$ where $R^T R = I$

notice $F_*(\alpha') = R\alpha'$ and $F_*(\alpha'') = R\alpha''$ thus, let $\bar{\alpha} = F \circ \alpha$,

$$\bar{T} = \bar{\alpha}' = (F \circ \alpha)' = F_* \alpha' = F_*(T)$$

$$\bar{N} = \frac{\bar{\alpha}''}{\|\bar{\alpha}''\|} = \frac{(F \circ \alpha)''}{\|(F \circ \alpha)''\|} = \frac{F_*(\alpha'')}{\|F_*(\alpha'')\|}$$

Thus, as $\|Rv\| = \|v\|$ for $R^T R = I$,

$$\bar{N} = \frac{F_*(\alpha'')}{\|\alpha''\|} = F_*\left(\frac{\alpha''}{\|\alpha''\|}\right) = F_*\left(\frac{T'}{\|T'\|}\right) = F_*(N).$$

Then,

$$\begin{aligned}
\bar{B} = \bar{T} \times \bar{N} &= F_*(T) \times F_*(N) \\
&= (RT) \times (RN) \\
&= \sum_{i=1}^{3} \left[(RT \times RN) \cdot e_i \right] e_i \\
&= \sum_{i=1}^{3} \det\left[RT \mid RN \mid e_i \right] e_i \\
&= \sum_{i=1}^{3} \det\left[RT \mid RN \mid RR^T e_i \right] e_i
\end{aligned}$$

P8S continued

$$\bar{B} = \sum_{i=1}^{3} \det[RT \mid RN \mid RR^T e_i]\, e_i$$

$$= \sum_{i=1}^{3} \det(R) \det(T \mid N \mid R^T e_i)\, e_i$$

$$= \sum_{i=1}^{3} \det(R) (T \times N) \cdot (R^T e_i)\, e_i$$

$$v \cdot (R^T w) = (R^T w) \cdot v = w^T R v = w^T (Rv) = w \cdot Rv = (Rv) \cdot w$$

$$= \det(R) \sum_{i=1}^{3} \left[(R(T \times N)) \cdot e_i \right] e_i$$

$$= \det(R)\, R(T \times N)$$

$$= \det(F_*)\, F_*(T \times N)$$

$$= \underline{\det(F_*)\, F_*(B)}.$$

Finally, we calculate $\bar{\kappa}$ and $\bar{\tau}$,

$$\bar{\kappa} = \| \bar{T}' \| = \| F_*(T') \| = \| RT' \| = \| T' \| = \kappa.$$

Likewise, as $\bar{N} = F_* N = RN$ and $\bar{B} = \det(F_*) R B$,

$$\bar{\tau} = -\bar{N} \cdot \bar{B}'$$

$$= -(RN) \cdot (\det(R) R B)'$$

$$= -\det(R) (RN)^T (RB')$$

$$= -\det(R)\, N^T R^T R B' \qquad (R^T R = I)$$

$$= -\det(R)\, N^T B'$$

$$= -\det(R)\, N \cdot B'$$

$$= -\det(R)(-\tau) \implies \boxed{\bar{\tau} = \det(F_*)\, \tau}$$

PROBLEM 85 continued

$\boxed{\text{(b.) Let } \alpha \text{ be unit-speed curve. Show } \alpha \text{ has constant curvature and torsion} \Leftrightarrow \alpha \text{ is helix.}}$

$\Leftarrow$ If $\alpha(s) = (R\cos(\gamma s), R\sin(\gamma s), m\gamma s)$ then

$\alpha'(s) = (-R\gamma\sin(\gamma s), R\gamma\cos(\gamma s), m\gamma)$ gives

us $\|\alpha'(s)\| = R^2\gamma^2 + m^2\gamma^2 = 1 \Leftrightarrow \gamma^2 = \frac{1-m^2}{R^2}$

$$\gamma^2(R^2+m^2) = 1 \Leftrightarrow \gamma = \frac{1}{\sqrt{R^2+m^2}}$$

So, $\alpha(s) = (R\cos(\gamma s), R\sin(\gamma s), m\gamma s)$ with $\gamma = \frac{1}{\sqrt{R^2+m^2}}$ gives unit-speed helix. We can calculate

$$\kappa = \|T'\| = \|\langle -R\gamma^2\cos\gamma s, -R\gamma^2\sin\gamma s, 0\rangle\| = \sqrt{R^2\gamma^4}$$

$$\therefore \boxed{\kappa = \frac{R}{R^2+m^2}}$$

Likewise,

$$B = T \times N = \langle -R\gamma\sin\theta, R\gamma\cos\theta, m\gamma\rangle \times \langle -\cos\theta, -\sin\theta, 0\rangle$$

$$= \langle m\gamma\sin\theta, -m\gamma\cos\theta, R\gamma\rangle \qquad \theta = \gamma s$$

$$B' = \langle m\gamma^2\cos\theta, m\gamma^2\sin\theta, 0\rangle$$

$$-B'\cdot N = m\gamma^2\cos^2\theta + m\gamma^2\sin^2\theta = m\gamma^2 = \frac{m}{R^2+m^2}$$

$$\therefore \boxed{\tau = \frac{m}{R^2+m^2}}$$

PROBLEM 85 continued

$\Rightarrow$] Suppose α has constant $\underbrace{\text{curvature}}_{\kappa}$ and $\underbrace{\text{torsion}}_{\tau}$.

If we can construct helix by selecting values for $m, R, \ \gamma = \dfrac{1}{\sqrt{R^2+m^2}}$ then that should suffice for the claim. Can we solve:

$$\kappa = \frac{R}{R^2+m^2} \quad \& \quad \tau = \frac{m}{R^2+m^2}$$

for m and R as functions of κ and τ?

$$\frac{\kappa}{\tau} = \frac{R/(R^2+m^2)}{m/(R^2+m^2)} = \frac{R}{m} \Rightarrow \kappa = \left(\frac{R}{m}\right)\tau$$

Well, better to say, $R = \dfrac{\kappa}{\tau} m$.

Likewise, $\kappa^2 + \tau^2 = \dfrac{R^2+m^2}{(R^2+m^2)^2} = \dfrac{1}{R^2+m^2}$

$$\boxed{\text{Hence, } \kappa = R(\kappa^2+\tau^2) \text{ and } \tau = m(\kappa^2+\tau^2)} \quad \underbrace{\gamma = \sqrt{\kappa^2+\tau^2}}$$

Therefore,

$$R = \frac{\kappa}{\kappa^2+\tau^2} \quad \text{and} \quad m = \frac{\tau}{\kappa^2+\tau^2}, \quad \gamma = \frac{1}{\sqrt{R^2+m^2}}$$

$$\underbrace{\tilde{\alpha}(s) = \left(\frac{\kappa}{\kappa^2+\tau^2}\cos(\gamma s), \ \frac{\kappa}{\kappa^2+\tau^2}\sin(\gamma s), \ \frac{\tau \gamma s}{\kappa^2+\tau^2} \right)}_{\text{this curve as curvature } \kappa \text{ and torsion } \tau}$$

hence by (a.) α is congruent $\tilde{\alpha}$, thus constant κ, τ curve congruent to helix.

$\boxed{P86}$ A_{ik} a p-form $\quad 1 \le i \le m$

B_{kj} a q-form $\quad 1 \le k \le r$

$\quad 1 \le j \le n$

$$(A \wedge B)_{ij} = \sum_{k=1}^{r} A_{ik} \wedge B_{kj} \quad / \quad (dA)_{ik} = dA_{ik}$$

$$\begin{aligned}
(d(A\wedge B))_{ij} &= d(A\wedge B)_{ij} \\
&= d\left(\sum_{k=1}^{r} A_{ik} \wedge B_{kj}\right) \\
&= \sum_{k=1}^{r}\left(dA_{ik} \wedge B_{kj} + (-1)^p A_{ik} \wedge dB_{kj}\right) \\
&= \left(dA \wedge B + (-1)^p A \wedge dB\right)_{ij}
\end{aligned}$$

Thus $\underline{d(A\wedge B) = dA \wedge B + (-1)^p A \wedge dB}$. //

If $A^T A = I$ for some matrix of functions then

$$\begin{aligned}
& d(A^T A) = dI \\
\Rightarrow\ & (dA^T) \wedge A + (-1)^0 A^T \wedge dA = 0 \\
\Rightarrow\ & (dA^T) A + A^T dA = 0 \\
\Rightarrow\ & -A A^T dA = A(dA^T) A \\
\therefore\ & \underline{-dA = A(dA^T)A}.
\end{aligned}$$

$\boxed{P87}$ $$\nabla_V W = \sum_{j=1}^{3} V[w^j]\, U_j$$

provided $W = w^1 U_1 + w^2 U_2 + w^3 U_3$

where $U_1 = \frac{\partial}{\partial x_1}$, $U_2 = \frac{\partial}{\partial x_2}$, $U_3 = \frac{\partial}{\partial x_3}$

(a.) $$\nabla_{fV} W = \sum_{j=1}^{3} (fV)[w^j]\, U_j$$

$$= f \sum_{j=1}^{3} V[w^j]\, U_j = f \nabla_V W .//$$

(b.) $$\nabla_V (fW) = \sum_{j=1}^{3} V[f w^j]\, U_j$$

$$= \sum_{j=1}^{3} \left(V[f]\, w^j + f V[w^j] \right) U_j$$

$$= V[f] \sum_{j=1}^{3} w^j U_j + f \sum_{j=1}^{3} V[w^j]\, U_j$$

$$= \underline{V[f]\, W + f\, \nabla_V W} .//$$

(c.) $$U[V \cdot W] = U\left[\sum_j v^j w^j\right]$$

$$= \sum_j \left(U[v^j]\, w^j + v^j\, U[w^j] \right)$$

$$= \underline{(\nabla_U V) \cdot W + V \cdot (\nabla_U W)} .//$$

(d.) $\|V\| = 1 \Rightarrow V \cdot V = 1$ hence by (c.)

we find $\underline{\nabla_V V = 0}$ .//

$\boxed{P88}$ Suppose $\{E_1, E_2, E_3\}$ is an orthonormal frame for $\mathbb{R}^3$ then $\exists$ fncts f^1, f^2, f^3 for $V \in \mathfrak{X}(\mathbb{R}^3)$ s.t. $V = f^1 E_1 + f^2 E_2 + f^3 E_3$ given by $f^i = V \cdot E_i$. Also, define $\omega_{ij}(p) \in (T_p\mathbb{R}^3)^*$ by $(\omega_{ij}(p))(v) = (\nabla_v E_i) \cdot E_j(p)$ for each $v \in T_p\mathbb{R}^3$.
If $V, W \in \mathfrak{X}(\mathbb{R}^3)$ where $V = \sum_i f^i E_i$ and $W = \sum_j g^j E_j$ then $\sum_j \omega_{ij}(V) E_j = \nabla_V E_i$ hence,

$$\nabla_V W = \sum_j \left(V[g^j] + \sum_i g^i \omega_{ij}(V) \right) E_j$$

(a.) If $A_{ij} = E_i \cdot U_j$ then $E_i = \sum_j A_{ij} U_j$

this is immediately true from general identity

$$V = \sum_j (V \cdot U_j) U_j \Rightarrow E_i = \sum_j (E_i \cdot U_j) U_j$$

$$\Rightarrow E_i = \sum_j A_{ij} U_j$$

//

Likewise,

$$\theta^i = \sum_{j=1}^3 \theta^i(U_j)\, dx^j$$

expand U_j in E-frame.

$$= \sum_j \theta^i \left(\sum_k (U_j \cdot E_k) E_k \right) dx^j$$

$$= \sum_j \sum_i \underbrace{E_k \cdot U_j}_{A_{kj}} \underbrace{\theta^i(E_k)}_{\delta_{ik}} dx^j = \sum_j A_{ij}\, dx^j.$$

P88 continued

$$\theta = \begin{bmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{bmatrix} \qquad d\xi = \begin{bmatrix} dx^1 \\ dx^2 \\ dx^3 \end{bmatrix} \hookrightarrow \theta = A\,d\xi$$

(b.) $$(\omega_{ij})(v) = (\nabla_v E_i) \cdot E_j$$

$$= \nabla_v \Big(\sum_k A_{ik} U_k \Big) \cdot E_j$$

$$= \sum_k \big(\nabla_v (A_{ik} U_k) \big) \cdot E_j$$

$$= \sum_k \big(v(A_{ik}) U_k + A_{ik} \cancel{\nabla_v U_k}^{\,0} \big) \cdot E_j$$

$$= \sum_k v(A_{ik})\, E_j \cdot U_k$$

$$= \sum_k v(A_{ik})\, A_{jk}$$

$$= \sum_k dA_{ik}(v)\, (A^T)_{kj}$$

$$= ((dA)\, A^T)_{ij}(v) \quad \therefore \quad \omega = (dA)\, A^T$$

$$\underline{\omega = dA \wedge A^T}.$$

(c.) $$\theta = A\,d\xi$$

$$d\theta = dA \wedge d\xi + A\, d(d\xi) \Rightarrow d\theta = dA \wedge A^T A \wedge d\xi$$

$$= (dA \wedge A^T) \wedge (A \wedge d\xi)$$

$$= \omega \wedge \theta \quad \therefore \underline{d\theta = \omega \wedge \theta}.$$

$$d\omega = d(dA \wedge A^T) = d(dA) \wedge A^T + (-1)^1 dA \wedge dA^T$$

$$\Rightarrow d\omega = -dA \wedge A^T A \wedge dA^T = (dA \wedge A^T) \wedge (-A \wedge dA^T)$$

$$= \underline{\omega \wedge \omega}.$$

P88 conclusion

$$dw = (dA \wedge A^T) \wedge (-A \wedge dA^T)$$
$$= w \wedge (-A \wedge dA^T) = w \wedge (dA \wedge A^T) = w \wedge w.$$

$$A^T A = I \Rightarrow dA^T \wedge A = -A^T \wedge dA$$
$$A A^T = I \Rightarrow \underline{dA \wedge A^T = -A \wedge dA^T}$$

$\boxed{P89}$ $\nabla_{\alpha'(s)} W = \frac{d}{ds}(W \circ \alpha)(s)$ let α be unit-speed

thus $v = 1$ and $T' = \kappa N$, $N' = -\kappa T + \tau B$, $B' = -\tau N$

thus consider, we propose $E_1|_\alpha = T$, $E_2|_\alpha = N$, $E_3|_\alpha = B$,

$$\begin{aligned} w_{12}(T) &= (\nabla_T E_1) \cdot E_2(\alpha(s)) \\ &= (\nabla_{\alpha'(s)} E_1) \cdot N \\ &= (E_1(\alpha(s)))'(s) \cdot N \\ &= T' \cdot N \\ &= \boxed{\kappa.} \end{aligned}$$

$$\begin{aligned} w_{13}(T) &= (\nabla_T E_1) \cdot E_3(\alpha(s)) \\ &= T' \cdot B \\ &= \kappa N \cdot B \\ &= \boxed{0.} \end{aligned}$$

$$\begin{aligned} w_{23}(T) &= (\nabla_T E_2) \cdot E_3(\alpha(s)) \\ &= N' \cdot B \\ &= (-\kappa T + \tau B) \cdot B \\ &= \boxed{\tau.} \end{aligned}$$

Remark: adapting frame to curve
$s \mapsto \alpha(s)$
gives one-dim'l tangent space, pointwise spanned by T. This is why we only look at $w_{ij}(T)$.

$\boxed{P90}$ $\phi = \sum f_i \theta^i$ where θ^i dual to E_i

Let's calculate $d\phi$,

$$d\phi = \sum_i \left(df_i \wedge \theta^i + f_i \, d\theta^i \right)$$

$$= \sum_i df_i \wedge \theta^i + f_i (\omega \wedge \theta)^i$$

$$= \sum_j df_j \wedge \theta^j + \sum_i f_i \sum_j \omega_{ij} \wedge \theta^j$$

$$= \underline{\sum_j \left(df_j + \sum_i f_i \, \omega_{ij} \right) \wedge \theta^j}\,.$$

$\boxed{P91}$ Define $S_p(v) = -(\nabla_v U)(p)$ for unit-normal U on $M \subseteq \mathbb{R}^3$

(a.) $S_p(cv+w) = -(\nabla_{cv+w} U)(p)$
$$= c(-\nabla_v U)(p) + (-\nabla_w U)(p)$$
$$= \underline{c\, S_p(v) + S_p(w)} . \quad \therefore S_p \text{ linear trans.}$$

Since $U \cdot U = 1$ we find that
$$v[U \cdot U] = v[1] = 0 = (\nabla_v U)\cdot U + U \cdot (\nabla_v U)$$
Thus $(-\nabla_v U)\cdot U = 0 \Rightarrow S_p(v)\cdot U = 0$

Hence $S_p(v) \in T_p M = U(p)^\perp$.

(b.) Symmetry of Shape operator, $S_p(v)\cdot w = v \cdot S_p(w)$

Consider $(u,v) \longmapsto X(u,v)$ the parametrization of M
then partial velocities $X_u = \frac{\partial X}{\partial u}$ and $X_v = \frac{\partial X}{\partial v}$ are $\perp$ to U.

Then $U \cdot X_u = 0 \Rightarrow 0 = \frac{\partial}{\partial v}[U \cdot X_u]$

thus $\frac{\partial U}{\partial v}\cdot X_u + U \cdot \frac{\partial X_u}{\partial v} = 0 \Rightarrow -S(X_v)\cdot X_u = -U\cdot X_{uv}$

Likewise $-S(X_u)\cdot X_v = -U \cdot X_{vu} = -S(X_v)\cdot X_u$

But $T_p M = \text{span}\{X_u, X_v\}$ thus, $v = aX_u + bX_v$
$w = cX_u + dX_v$

$$S_p(v)\cdot w = (aS(X_u) + bS(X_v))\cdot(cX_u + dX_v)$$
$$= ac\, S(X_u)\cdot X_u + (ad+bc)\, S(X_u)\cdot X_v + bd\, S(X_v)\cdot X_v$$

Likewise,
$$v \cdot S_p(w) = (aX_u + bX_v)\cdot(cS(X_u) + dS(X_v))$$
$$= ac\, X_u \cdot S(X_u) + (ad+bc)\, X_v \cdot S(X_u) + bd\, X_v \cdot S(X_v)$$

Thus $v \cdot S_p(w) = S_p(v)\cdot w$. //

P91 continued

(c.) Show $S(v) \times S(w) = K\, v \times w$

and $S(v) \times w + v \times S(w) = 2H\, v \times w$

where $K = \det(S)$ and $H = \text{trace}(S)$

I should clarify, we need v, w LI in T_pM,

Then $S(v) = av + bw$ and $S(w) = cv + dw$

$$S(v) \times S(w) = (av+bw) \times (cv+dw)$$
$$= (ac - bc)\, v \times w$$
$$= \det(S)\, v \times w$$
$$= \underline{K\, v \times w}.$$

$$[S] = \underbrace{\begin{bmatrix} a & c \\ b & d \end{bmatrix}}_{\substack{\text{matrix of } S \\ \text{w.r.t. } v,w \\ \text{basis}}}$$

Likewise,

$$S(v) \times w + v \times S(w) =$$
$$= (av + bw) \times w + v \times (cv + dw)$$
$$= (a + d)\, v \times w$$
$$= \text{trace}(S)\, v \times w$$
$$= \underline{H\, v \times w}.//$$

$\text{trace}(S) = \text{trace}[S] = a + d$

(d.) Given $S_p(v_1) = 6v_1$ and $S_p(v_2) = 7v_2$

then v_1, v_2 are LI and thus

the matrix of S_p with respect to v_1, v_2

is diagonal; $[S_p] = \begin{bmatrix} 6 & 0 \\ 0 & 7 \end{bmatrix}$

$\therefore$ $\boxed{K(p) = 42}$ and $\boxed{H(p) = 13}$

P92 Use E, F, G, L, M, N to calculate K & H for the Helicoid $\mathbf{X}(u,v) = \langle u\cos v, u\sin v, bv\rangle$ $b \neq 0$ and Cylinder $\mathbf{X}(u,v) = \langle R\cos u, R\sin u, v\rangle$, $R > 0$ constant.

$$E = \mathbf{X}_u \cdot \mathbf{X}_u \qquad L = U \cdot \mathbf{X}_{uu}$$

$$F = \mathbf{X}_u \cdot \mathbf{X}_v \qquad M = U \cdot \mathbf{X}_{uv} \qquad U = \mathbf{X}_u \times \mathbf{X}_v$$

$$G = \mathbf{X}_v \cdot \mathbf{X}_v \qquad N = U \cdot \mathbf{X}_{vv}$$

$$K = \frac{LN - M^2}{EG - F^2} \qquad H = \frac{GL + EN - 2FM}{2(EG - F^2)}$$

(a.) $\mathbf{X}_u = \langle \cos v, \sin v, 0\rangle$ $\qquad E = 1$

$\mathbf{X}_v = \langle -u\sin v, u\cos v, b\rangle$ $\qquad F = 0$

$U = \dfrac{\mathbf{X}_u \times \mathbf{X}_v}{\|\mathbf{X}_u \times \mathbf{X}_v\|} = \dfrac{\langle b\sin v, -b\cos v, u\rangle}{\sqrt{u^2+b^2}}$ $\qquad G = u^2 + b^2$

$\mathbf{X}_{uu} = \langle 0, 0, 0\rangle$ $\qquad L = 0$

$\mathbf{X}_{uv} = \langle -\sin v, \cos v, 0\rangle$ $\qquad M = -b/\sqrt{u^2+b^2}$

$\mathbf{X}_{vv} = \langle -u\cos v, -u\sin v, 0\rangle$ $\qquad N = 0$

Thus,

$$K = \frac{0 - (-b)^2/u^2+b^2}{1(u^2+b^2) - 0^2} \Rightarrow \boxed{K = \frac{-b^2}{(u^2+b^2)^2}}.$$

$$H = \frac{(u^2+b^2)(0) + 1(0) - 2(0)(-b)}{2(1\cdot(u^2+b^2) - 0^2)} \Rightarrow \boxed{H = 0}.$$

<u>Remark</u>: don't forget to normalize $\mathbf{X}_u \times \mathbf{X}_v$ to find the <u>unit</u> normal U.

P92 continued

(b.) $\mathbf{X}(u,v) = \langle R\cos u,\ R\sin u,\ v \rangle \quad , \quad R \neq 0$

$$\mathbf{X}_u = \langle -R\sin u,\ R\cos u,\ 0 \rangle$$

$$\mathbf{X}_v = \langle 0,\ 0,\ 1 \rangle$$

$$\tilde{U} = \mathbf{X}_u \times \mathbf{X}_v = \langle R\cos u,\ R\sin u,\ 0 \rangle$$

oops, need to normalize.

$$U = \frac{\tilde{U}}{R}$$

$$\mathbf{X}_{uu} = \langle -R\cos u,\ -R\sin u,\ 0 \rangle$$

$$\mathbf{X}_{uv} = \langle 0,\ 0,\ 0 \rangle$$

$$\mathbf{X}_{vv} = \langle 0, 0, 0 \rangle$$

$$E = \mathbf{X}_u \cdot \mathbf{X}_u = R^2$$

$$F = \mathbf{X}_u \cdot \mathbf{X}_v = 0$$

$$G = \mathbf{X}_v \cdot \mathbf{X}_v = 1$$

$$L = U \cdot \mathbf{X}_{uu} = -R^2/R = -R$$

$$M = U \cdot \mathbf{X}_{uv} = 0$$

$$N = U \cdot \mathbf{X}_{vv} = 0$$

Thus,

$$K = \frac{LN - M^2}{EG - F^2} = \frac{0 - 0}{R^2 - 0} = 0 \quad \therefore \quad \boxed{K = 0}$$

$$H = \frac{GL + EN - 2FM}{2(EG - F^2)} = \frac{-R + 0 - 0}{2(R^2 - 0)} \Rightarrow \boxed{H = \frac{-1}{2R}}$$