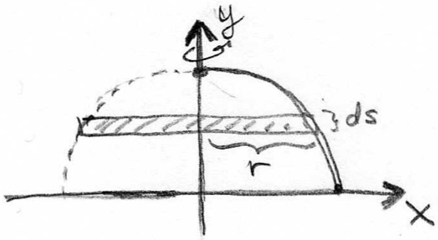
Solutions to select problems on Applications ($\S 9.2$ # 14, 5; $\S 6.5$ # 2, 7, 8, 20)

§ 9.2 #14 Find surface area of volume of revolution formed from $y = 1 - x^2$ for $0 \le x \le 1$ rotated around y-axis.



$$dA = 2\pi r \, ds$$

We can use x to parametrize ds. This is nice as then $r = x$

and $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

Hence, integrate the following $\left(\frac{dy}{dx} = -2x\right)$

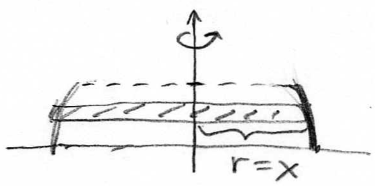
$$S = \int_0^1 2\pi x \sqrt{1 + 4x^2}\, dx$$

$$= \int_1^5 \frac{2\pi}{8} \sqrt{u}\, du \qquad \begin{array}{ll} u = 1 + 4x^2 & u(1) = 5 \\ du = 8x\,dx & u(0) = 1 \end{array}$$

$$= \frac{\pi}{4} \frac{2}{3} u^{3/2} \Big|_1^5$$

$$= \boxed{\frac{\pi}{6}\left(\sqrt{125} - 1\right)}$$

§ 9.2 #15 Find surface area generated from revolving $x = \sqrt{a^2 - y^2}$ for $0 \le y \le a/2$ around the y-axis



$$dA = 2\pi r \, ds$$
$$= 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Need to find y as function of x,

We note $x = \sqrt{a^2 - y^2} \Rightarrow x^2 = a^2 - y^2 \Rightarrow y = \sqrt{a^2 - x^2}$ since by the picture we want $y \ge 0$. Note $\frac{dy}{dx} = \frac{-x}{\sqrt{a^2 - x^2}}$ thus $(dy/dx)^2 = \frac{x^2}{a^2 - x^2}$. Finally, note that $0 \le y \le a/2 \Rightarrow \sqrt{a^2 - a^2/4} \le x \le a \Rightarrow \frac{\sqrt{3}}{2} a \le x \le a$. Hence,

$$S = \int_{\frac{\sqrt{3}a}{2}}^{a} 2\pi x \sqrt{1 + \frac{x^2}{a^2 - x^2}}\, dx$$

integration ↴

§9.2 #15 continued

$$S = \int_{\frac{\sqrt{3}a}{2}}^{a} 2\pi x \sqrt{1 + \frac{x^2}{a^2-x^2}}\, dx$$

$$= \int_{\frac{\sqrt{3}a}{2}}^{a} 2\pi x \sqrt{\frac{a^2-x^2+x^2}{a^2-x^2}}\, dx$$

$$= \int_{\sqrt{3}a/2}^{a} \frac{2\pi a\, x\, dx}{\sqrt{a^2-x^2}} \longrightarrow$$

let $u = a^2 - x^2$
$du = -2x\,dx$
$u\left(\frac{\sqrt{3}a}{2}\right) = a^2/4$
$u(a) = 0$

$$= \int_{a^2/4}^{0} \frac{-\pi a\, du}{\sqrt{u}}$$

$$= -\pi a \left(2u^{1/2}\Big|_{a^2/4}^{0}\right)$$

$$= -\pi a \left(2\sqrt{0} - 2\sqrt{a^2/4}\right)$$

$$= \boxed{\pi a^2}$$ neat.

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§6.5 #2) find average of $f(x) = \sin(4x)$ on $[-\pi, \pi]$.

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(4x)dx = \boxed{0.}$$

$(a = -\pi,\ b = \pi)$ (odd function!)

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§6.5 #7) find average of $h(x) = \cos^4(x)\sin(x)$, $[0, \pi]$

$$h_{avg} = \frac{1}{\pi}\int_0^{\pi} \cos^4(x)\sin(x)dx$$

$u = \cos(x)$ $u(0) = 1$
$du = -\sin(x)dx$ $u(\pi) = -1$

$$= \frac{1}{\pi}\int_1^{-1} u^4(-du)$$

$$= \frac{1}{\pi}\left(\frac{u^5}{5}\Big|_{-1}^{1}\right)$$

$$= \boxed{\frac{2}{5\pi}}$$

§6.5#8 Let $h(r) = 3/(1+r)^2$ on $[1, 6]$,

$$h_{avg} = \frac{1}{6-1}\int_1^6 \frac{3\,dr}{(1+r)^2}$$

$u = 1+r$
$du = dr$
$u(1) = 2$
$u(6) = 7$

$$= \frac{1}{5}\int_2^7 \frac{3\,du}{u^2}$$

$$= \frac{3}{5}\left(\frac{-1}{u}\Big|_2^7\right)$$

$$= \frac{3}{5}\left(\frac{-1}{7} + \frac{1}{2}\right)$$

$$= \frac{3}{5}\left(\frac{-2+7}{14}\right)$$

$$= \boxed{\frac{3}{14}}$$

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§6.5#20 Given $s = \frac{1}{2}gt^2$ compare the average velocity w.r.t time verses the average velocity w.r.t. s.

$$\frac{ds}{dt} = gt \quad \text{and} \quad t \geq 0 \quad \text{thus} \quad t = \sqrt{\frac{2s}{g}}$$

So we have $V(t) = gt$ while $V(s) = g\sqrt{\frac{2s}{g}} = \sqrt{2gs}$

$$\overline{V(t)} = \frac{1}{T-0}\int_0^T gt\,dt = \frac{1}{T}\left(\frac{1}{2}gT^2\right) = \underline{\frac{1}{2}gT = \frac{1}{2}V_T}.$$

$$\overline{V(s)} = \frac{1}{\frac{1}{2}gT^2 - 0}\int_0^{\frac{1}{2}gT^2} \sqrt{2gs}\,ds$$

$$= \frac{2\sqrt{2g}}{gT^2}\,\frac{2}{3}\,s^{3/2}\Big|_0^{\frac{1}{2}gT^2}$$

$$= \frac{4\sqrt{2}}{\sqrt{g}\,T^2}\,\frac{1}{3}\left(\frac{1}{2}gT^2\right)^{3/2}$$

$$= \frac{2\cancel{\sqrt{8}}}{\cancel{\sqrt{g}}\,\cancel{T^2}}\,\frac{1}{3}\,\frac{1}{\cancel{\sqrt{8}}}\,g\cancel{\sqrt{g}}\,T^3$$

$$= \underline{\frac{2}{3}gT = \frac{2}{3}V_T}.$$

<u>Remark:</u> this is <u>not</u> test worthy. It involves ambiguities of notation I have not elaborated on in lecture.